Selçuk J. Appl. Math.
Vol. 9. No. 2, pp. 9 – 18 , 2008

Selçuk Journal of Applied Mathematics

Some *I*-related properties of triple sequences

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Abstract. Ideal convergence was presented by Kostyrko et al. in 2001. This concept was extended to the double sequences by Tripathy et al. in 2006. In this paper we introduce the notions of $\mathbf{I}-$convergence, $\mathbf{I}-$bounded, $\mathbf{I}-$inferior and $\mathbf{I}-$superior for triple sequences. We also investigate some further properties of $\mathbf{I}-$limit superior and $\mathbf{I}-$limit inferior of triple sequences.

Key words: Double sequence; triple sequence; statistical convergence; I-convergence; double natural density; triple natural density
2000 Mathematics Subject Classification. 40A05, 26A03.

1. Preliminaries

In this article we aimed to extend the notion of statistically convergent triple sequences to $\mathbf{I}-$ convergent triple sequences. Now we recall some definitions and notions introduced in [14].

Definition 1. *A function* $x : \mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{R}(\mathbb{C})$ *is called a real (complex) triple sequence.*

Definition 2. *A triple sequence* (x_{nkl}) *is said to be convergent to* L *in Pringsheim's sense if for every* $\varepsilon>0$, *there exists* $n_0(\varepsilon)\in\mathbb{N}$ *such that* $|x_{nkl}-L|<\varepsilon$ whenever $n,k,l\geq n_0$.

Definition 3. *A triple sequence* (x_{nkl}) *is said to be Cauchy sequence if for every* $\varepsilon > 0$, *there exists* $n_0(\varepsilon) \in \mathbb{N}$ *such that*
$|x_{pqr} - x_{nkl}| < \varepsilon$ whenever $p \geq n \geq n_0, q \geq k \geq n_0, r \geq l \geq n_0$.

Definition 4. *A triple sequence* (x_{nkl}) *is said to be bounded if there exists* $M > 0$ *such that* $|x_{nkl}| < M$ *for all* $n,k,l \in \mathbb{N}$.
We denote the set of all bounded triple sequences by ℓ_∞^3. It can easily be shown that ℓ_∞^3 is a normed space by

$$\|x\|_{(\infty,3)} = \sup_{n,k,l} |x_{nkl}| < \infty.$$

The notion of statistically convergent double sequences was introduced by Tripathy [12]. Recall that a subset E of $\mathbb{N} \times \mathbb{N}$ is said to have density $\rho(E)$ if $\rho(E) = \lim_{p,q \to \infty} \frac{1}{pq} \sum_{n \leq p} \sum_{k \leq q} \chi_E(n,k)$ exists, where $\chi_E(n,k)$ is the characteristic function of the set E. Thus a double sequence (x_{nk}) is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\rho(\{(n,k) \in \mathbb{N} \times \mathbb{N} \ : \ |x_{nk} - L| \geq \varepsilon\}) = 0$$

[13]. The notion of statistically convergent double sequences was extended to $\mathbf{I}-$ convergent double sequences by Tripathy in [13].
The notion of statistically convergent triple sequences was introduced by Sahiner [14]. Recall that a subset A of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is said to have density $\rho(A)$ if

$$\rho(A) = \lim_{p,q,r \to \infty} \frac{1}{pqr} \sum_{n \leq p} \sum_{k \leq q} \sum_{l \leq r} \chi_A(n,k,l)$$

exists. For example if $A = \{(n^3, k^3, l^3) \ : \ n,k,l \in \mathbb{N}\}$ then

$$\rho(A) = \lim_{p,q,r} \frac{|K(p,q,r)|}{pqr} \leq \lim_{p,q,r} \frac{\sqrt[3]{p}\sqrt[3]{q}\sqrt[3]{r}}{pqr} = 0$$

where

$$K(p,q,r) = \{(n,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \ : \ p \geq n, \ q \geq k, \ r \geq l\}$$

and $|K(p,q,r)|$ is the cardinality of $K(p,q,r)$. Thus a triple sequence (x_{nkl}) is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0$,

$$\rho(\{(n,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \ : \ |x_{nkl} - L| \geq \varepsilon\}) = 0.$$

Let X be a non-empty set, then a non-void class $\mathtt{I} \subseteq 2^X$ (power set of X) is called an *ideal* if $\mathtt{I}$ is additive (i.e. $A, B \in \mathtt{I} \Rightarrow A \cup B \in \mathtt{I}$) and hereditary (i.e. $A \in \mathtt{I}$ and $B \subseteq A \Rightarrow B \in \mathtt{I}$). An ideal $\mathtt{I} \subseteq 2^X$ is said to be non-trival if $\mathtt{I} \neq 2^X$.
A non-trival ideal $\mathtt{I}$ is said to be *admissible* if $\mathtt{I}$ contains every finite subset of $\mathbb{N}$. A non-trival ideal $\mathtt{I}$ is said to be *maximal* if there does not exist any non trival ideal $\mathtt{J} \neq \mathtt{I}$ containing $\mathtt{I}$ as a subset.
In this article we aimed to introduce and examine $\mathtt{I}-$ related interesting properties of triple sequences.
We denote the ideals of $2^{\mathbb{N}}$ by $\mathtt{I}$; the ideals of $2^{\mathbb{N}\times\mathbb{N}}$ by $\mathtt{I}_2$ and the ideals of $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ by $\mathtt{I}_3$.

2. Ideal convergence of triple sequences

Definition 5. *Let* $\mathtt{I}_3$ *be an ideal of* $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$, *then a triple sequence* (x_{nkl}) *is said to be* $\mathtt{I}-$ *convergent to* L *in Pringsheim's sense if for every* $\varepsilon > 0$,

$$\{(n,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \ : \ |x_{nkl} - L| \geq \varepsilon\} \in \mathtt{I}_3.$$

If (x_{nkl}) *is* $\mathtt{I}-$ *convergent to* L *we write* $\mathtt{I}_3$ - $\lim x_{nkl} = L$.

Now we give some examples of ideals and corresponding $I-$convergences.

(I) Let $\mathtt{I}_3(f)$ be the family of all finite subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Then $\mathtt{I}_3(f)$ is an admissible ideal in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ and $\mathtt{I}_3(f)$ convergence coincides with the convergence of triple sequences in Pringsheim's sense.
(II) Let $A \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ be a three dimensional set of positive integers and let $A(p,q,r)$ be the cardinality of (n,k,l) in A such that $n \leq p$, $k \leq q$, $l \leq r$. In case of the sequence $\lim\limits_{p,q,r}\left(\frac{A(p,q,r)}{pqr}\right)$ has a limit in Pringsheim's sense then we say that A has a triple natural density

and we denote this by $\lim_{p,q,r}\left(\frac{A(p,q,r)}{pqr}\right)=\rho(A)$. Put $\mathtt{I}_3(\rho)=\{A\subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \rho(A)=0\}$. Then $\mathtt{I}_3(\rho)$ is an admissible ideal in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $\mathtt{I}_3(\rho)$ convergence coincides with the statistical convergence in Pringsheim's sense [14].

Example 1. *Let* $\mathtt{I}=\mathtt{I}_3(\rho)$. *Define the triple sequence* (x_{nkl}) *by*

$$x_{nkl}=\begin{cases}1 & \text{, if } n,k \text{ and } l \text{ are squares}\\ 5 & \text{, otherwise.}\end{cases}$$

Then for every $\varepsilon>0$

$$\rho(\{(n,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{nkl}-5|\geq\varepsilon\})\leq\lim_{p,q,r}\frac{\sqrt{p}\sqrt{q}\sqrt{r}}{pqr}=0.$$

This implies that $I-\lim x_{nkl}=5$ *in Pringsheim's sense. But, the sequence* (x_{nkl}) *is not convergent to* 5 *in Pringsheim's sense.*

Remark 1. If $\mathtt{I}_3$ *is admissible and* (x_{nkl}) *converges to* L *in Pringsheim's sense, then* (x_{nkl}) *is* $\mathtt{I}-$*convergent to* L *in Pringsheim's sense.*

3. I- limit superior and I- limit inferior for triple sequences

Definition 6. *Let* $\mathtt{I}_3$ *be an ideal of* $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$. *A number* ξ *is said to be an* $\mathtt{I}_3-$*limit point of the triple sequence* (x_{nkl}) *provided that there exists a set* $M=\{n_1<n_2<...\}\times\{k_1<k_2<...\}\times\{l_1<l_2<...\}\subset\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ *such that* $M\notin\mathtt{I}_3$ *and* $P-\lim x_{n_i k_j l_m}=\xi$ *for all* $i,j,m=1,2,...$.

Definition 7. *A number* ζ *is called to be an* $\mathtt{I}-$*cluster point of the triple sequence* (x_{nkl}) *if for each* $\varepsilon>0$,

$$\{(n,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{nkl}-\zeta|<\varepsilon\}\notin\mathtt{I}_3.$$

Definition 8. *A real triple sequence* (x_{nkl}) *is said to be bounded if there is a* $K>0$ *such that* $\{(n,k,l)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{nkl}|>K\}\in\mathtt{I}_3$.

Now let (x_{nkl}) be a triple sequence and $t \in \mathbb{R}$. Then we set the following sets to be able to give the definitions of $\mathtt{I}-\liminf x$ and $\mathtt{I}-\limsup x$ of (x_{nkl}).

$$M_t = \{(n,k,l) \; : \; x_{nkl} > t\}, \; M^t = \{(n,k,l) \; : \; x_{nkl} < t\}.$$

Definition 9. *(a) If there is a* $t \in \mathbb{R}$ *such that* $M_t \notin \mathtt{I}_3$, *we put*

$$\mathtt{I}-\limsup x = \sup\{t \in \mathbb{R} \; : \; M_t \notin \mathtt{I}_3\}.$$

If $M_t \in \mathtt{I}_3$ *holds for each* $t \in \mathbb{R}$ *then we put* $\mathtt{I}-\limsup x = -\infty.$

(b)If there is a $t \in \mathbb{R}$ *such that* $M^t \notin \mathtt{I}_3$, *we put*

$$\mathtt{I}-\liminf x = \inf\{t \in \mathbb{R} \; : \; M^t \notin \mathtt{I}_3\}.$$

If $M^t \in \mathtt{I}_3$ *holds for each* $t \in \mathbb{R}$ *then we put* $\mathtt{I}-\liminf x = +\infty.$

Example 2. *If we define* (x_{nkl}) *by*

$$x_{nkl} = \left\{ \begin{array}{lll} n & , & n \text{ is an odd square} \\ 2 & , & n \text{ is an even square} \\ 1 & , & n \text{ is an odd nonsquare} \\ 0 & , & n \text{ is an even nonsquare} \end{array} \right\}$$

or

$$x_{nkl} = \left\{ \begin{array}{lll} k & , & k \text{ is an odd square} \\ 2 & , & k \text{ is an even square} \\ 1 & , & k \text{ is an odd nonsquare} \\ 0 & , & k \text{ is an even nonsquare} \end{array} \right\}$$

or

$$x_{nkl} = \left\{ \begin{array}{lll} l & , & l \text{ is an odd square} \\ 2 & , & l \text{ is an even square} \\ 1 & , & l \text{ is an odd nonsquare} \\ 0 & , & l \text{ is an even nonsquare} \end{array} \right\}$$

then, in each case, (x_{nkl}) *is not bounded above but it is* $\mathtt{I}-$ *bounded. Also,* $\{t \in \mathbb{R} \; : \; M_t \notin \mathtt{I}_3\} = (-\infty, 1)$, $\{t \in \mathbb{R} \; : \; M^t \notin \mathtt{I}_3\} = (0, \infty)$ *and thus* $\mathtt{I}-\limsup x = 1$, $\mathtt{I}-\liminf x = 0$. *On the other hand* (x_{nkl}) *can not be* $\mathtt{I}-$ *convergent in Pringsheim's sense and the set of* $\mathtt{I}-$

cluster points in Pringsheim's sense is $\{0,1\}$. *So we have the following.*

Theorem 1. *(i)* $\beta = \mathbf{I} - \limsup x$ *if and only if for each* $\varepsilon > 0$,

$$\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \; x_{nkl} > \beta - \varepsilon\} \notin \mathbf{I}_3$$

and

$$\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \; x_{nkl} > \beta + \varepsilon\} \in \mathbf{I}_3$$

(ii) $\alpha = \mathbf{I} - \liminf x$ *if and only if for each* $\varepsilon > 0$,

$$\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \; x_{nkl} < \alpha + \varepsilon\} \notin \mathbf{I}_3$$

and

$$\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \; x_{nkl} < \alpha - \varepsilon\} \in \mathbf{I}_3.$$

Proof. (i) We prove necessity first. Let $\varepsilon > 0$ be given. Since $\beta + \varepsilon > \beta$, $(\beta + \varepsilon) \notin \{t \; : \; M_t \notin \mathbf{I}_3\}$ and $\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \; x_{nkl} > \beta + \varepsilon\} \in \mathbf{I}_3$. Similarly, since $\beta - \varepsilon < \beta$, there exists some t' such that $\beta - \varepsilon < t' < \beta$ and $t' \in \{t \; : \; M_t \notin \mathbf{I}_3\}$. Thus $\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \; x_{nkl} > t'\} \notin \mathbf{I}_3$ and $\{(n,k,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \; x_{nkl} > \beta - \varepsilon\} \notin \mathbf{I}_3$.

Now we prove sufficiency. If $\varepsilon > 0$ then $(\beta + \varepsilon) \notin \{t \; : \; M_t \notin \mathbf{I}_3\}$ and $\mathbf{I}_3 - \limsup x_{nkl} \leq \beta + \varepsilon$. On the other hand we already have $\mathbf{I}_3 - \limsup x_{nkl} \geq \beta - \varepsilon$ and this means $\mathbf{I}_3 - \limsup x_{nkl} = \beta$ as desired.

(ii) can be proved analogously.

Theorem 2. *For every real triple sequence* (x_{nkl}), $\mathbf{I}_3 - \liminf x_{nkl} \leq \mathbf{I}_3 - \limsup x_{nkl}$.

Proof. If (x_{nkl}) is any real triple sequence we have three possibilities:

(1) The case $\mathbf{I}_3 - \limsup x_{nkl} = +\infty$ is clear.

(2) If $\mathbf{I}_3 - \limsup x_{nkl} = -\infty$. Then we have

$$t \in \mathbb{R} \Rightarrow M_t \in \mathbf{I}_3 \text{ and } M^t \notin \mathbf{I}_3.$$

Thus, $\mathbf{I}_3 - \liminf x_{nkl} = \inf\{t \; : \; M^t \notin \mathbf{I}_3\} = \inf \mathbb{R} = -\infty$ and $\mathbf{I}_3 - \liminf x_{nkl} \leq \mathbf{I}_3 - \limsup x_{nkl}$.

(3) If $-\infty < \mathtt{I}_3 - \limsup x_{nkl} < +\infty$ and if $\beta = \mathtt{I}_3 - \limsup x_{nkl}$ then for any $t \in \mathsf{R}$,

$$\beta < t \Rightarrow M_t \in \mathtt{I}_3 \text{ and } M^t \notin \mathtt{I}_3.$$

But this means $\mathtt{I}_3 - \liminf x_{nkl} = \inf\{t : M^t \notin \mathtt{I}_3\} \le \beta.$

Theorem 3. *For any* $\mathtt{I}-$ *bounded real triple sequence* (x_{nkl}) *we have the following inequalities.*

$$P - \liminf x_{nkl} \le \mathtt{I}_3 - \liminf x_{nkl} \le \mathtt{I}_3 - \limsup x_{nkl} \le P - \limsup x_{nkl}.$$

Proof. The case $P - \limsup x_{nkl} = +\infty$ is straightforward. Let $P - \limsup x_{nkl} = L < +\infty$. Then for any $t' > L,$ $M_{t'} \in \mathtt{I}_3.$ So $t' \notin \{t : M_t \notin \mathtt{I}_3\}$ implies $\mathtt{I}_3 - \limsup x_{nkl} = \sup\{t : M_t \notin \mathtt{I}_3\} < t'$ and $\mathtt{I}_3 - \limsup x_{nkl} \le L.$ This proves the last inequality. For the first one, if $P - \liminf x_{nkl} = -\infty$ then clearly the inequality holds. Let $P - \liminf x_{nkl} = T > -\infty$. Then for any $t' < T,$ $M^{t'} \in \mathtt{I}_3.$ So $t' \notin \{t : M^t \notin \mathtt{I}_3\}$ implies $\mathtt{I}_3 - \liminf x_{nkl} = \sup\{t : M^t \notin \mathtt{I}_3\} > t'$ and $\mathtt{I}_3 - \limsup x_{nkl} \ge T.$

Remark 2. *If* $\mathtt{I}_3 - \lim x_{nkl}$ *exists then* (x_{nkl}) *is* $\mathtt{I}-$ *bounded.*

Remark 3. *Note that ideal boundedness of triple sequences implies that* $\mathtt{I}_3 - \limsup$ *and* $\mathtt{I}_3 - \liminf$ *are finite.*

Recall that the core of a bounded double sequence x_{nk}, that is, $P - core(x_{nk}),$ is the interval $[P - \liminf x_{nk}, P - \limsup x_{nk}] = P - core(x_{nk});$ $\mathtt{I}-$ core of bounded double sequence x_{nk} is the interval $[\mathtt{I}_2 - \liminf x_{nk}, \mathtt{I}_2 - \limsup x_{nk}].$ Analogously we give the definitions of $P - core$ and $\mathtt{I} - core$ of a bounded triple sequence x_{nkl} .

Definition 10. *We define the* $P - core$ *of bounded real triple sequence* x_{nkl} *by*

$$[P - \liminf x_{nkl}, P - \limsup x_{nkl}].$$

Definition 11. *If* (x_{nkl}) *is any* $\mathtt{I}_3$ *bounded real triple sequence then we define its* $\mathtt{I}-core$ *by*

$$\left[\mathtt{I}_3-\liminf x_{nkl}, \mathtt{I}_3-\limsup x_{nkl}\right].$$

We use $\mathtt{I}_3-core(x_{nkl})$ *to denote* $\mathtt{I}_3$ *core of* x_{nkl} .

Corollary 1. *If* (x_{nkl}) *is any real triple sequence then we have*

$$\mathtt{I}_3-core(x_{nkl})\subset P-core(x_{nkl}).$$

Theorem 4. *A real triple sequence* (x_{nkl}) *is* $\mathtt{I}_3-$ *convergent if and only if* $\mathtt{I}_3-\liminf x_{nkl}=\mathtt{I}_3-\limsup x_{nkl}$.

Proof. Let $L=\mathtt{I}_3-\lim x_{nkl}$. Then $\{(n,k,l)\in \mathrm{N}\times\mathrm{N}\times\mathrm{N}:\ x_{nkl}>L+\varepsilon\}\in\mathtt{I}_3$ and $\{(n,k,l)\in \mathrm{N}\times\mathrm{N}\times\mathrm{N}:\ x_{nkl}<L-\varepsilon\}\in\mathtt{I}_3$. Then for any $t\geq L+\varepsilon$ and $t'<L-\varepsilon$, the sets M_t and $M^{t'}$ are in $\mathtt{I}_3$. Hence $\sup\{t\ :\ M_t\notin\mathtt{I}_3\}\leq L+\varepsilon$ and $\inf\{t'\ :\ M^{t'}\notin\mathtt{I}_3\}\geq L-\varepsilon$. To prove sufficiency let $\varepsilon>0$ and $L=\mathtt{I}_3-\liminf x_{nkl}=\mathtt{I}_3-\limsup x_{nkl}$. Then since

$$\{(n,k,l)\in \mathrm{N}\times\mathrm{N}\times\mathrm{N}:\ |x_{nkl}-L|\geq\varepsilon\}\subseteq\{(n,k,l)\in \mathrm{N}\times\mathrm{N}\times\mathrm{N}:\ x_{nkl}>L+\varepsilon\}$$
$$\cup\{(n,k,l)\in \mathrm{N}\times\mathrm{N}\times\mathrm{N}:\ x_{nkl}<L-\varepsilon\}$$

we conclude that $L=\mathtt{I}_3-\lim x_{nkl}$.

Note that if (x_{nkl}) is a bounded real triple sequence then we denote the set of all $\mathtt{I}_3-$ cluster points of (x_{nkl}) by $\mathtt{I}_3(\Gamma_x)$.

Theorem 5. *Suppose* (x_{nkl}) *is a bounded real triple sequence then*

$$\mathtt{I}_3-\limsup x_{nkl}=\max\mathtt{I}_3(\Gamma_x)$$

and

$$\mathtt{I}_3-\liminf x_{nkl}=\min\mathtt{I}_3(\Gamma_x).$$

Proof. Let $\mathtt{I}_3-\limsup x_{nkl}=L=\sup\{t\ :\ \{(n,k,l)\in \mathrm{N}\times\mathrm{N}\times\mathrm{N}:\ x_{nkl}>t\}\notin\mathtt{I}_3\}$. If $L'>L$ then there exists some $\varepsilon>0$ such that

$\{(n,k,l)\in \mathbb{N}:\ x_{nkl} > L'-\varepsilon\}\in \mathbf{I}_3$ and this means there exists some $\varepsilon>0$ such that

$$\{(n,k,l)\in \mathbb{N}\times\mathbb{N}\times\mathbb{N}:\ |x_{nkl}-L'|<\varepsilon\}\in \mathbf{I}_3,$$

that is, $L'\notin(\Gamma_x)$.

Now, we show L is really an $\mathbf{I}_3-$ cluster point of (x_{nkl}). Clearly, for each $\varepsilon>0$ there exists some $t\in(L-\varepsilon, L+\varepsilon)$ such that $\{(n,k,l)\in \mathbb{N}\times\mathbb{N}\times\mathbb{N}:\ x_{nkl}>t\}\notin \mathbf{I}_3$ and this implies $\{(n,k,l)\in \mathbb{N}\times\mathbb{N}\times\mathbb{N}:\ |x_{nkl}-L|<\varepsilon\}\notin \mathbf{I}_3$.

4. Further properties

In this section we prove some further results on $\mathbf{I}-\limsup$ and $\mathbf{I}-\liminf$ of a triple sequence.

Theorem 6. *Let* $\mathbf{I}_3$ *be an ideal of* $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$. *If* $x=(x_{nkl}), y=(y_{nkl})$ *are two* $\mathbf{I}-$ *bounded triple sequences in Pringsheim's sense, then*

(i) $\mathbf{I}-\limsup(x+y)\le \mathbf{I}-\limsup x+\mathbf{I}-\limsup y$

(ii) $\mathbf{I}-\liminf(x+y)\ge \mathbf{I}-\liminf x+\mathbf{I}-\liminf y$.

Proof. Since the proof of (ii) is analogous we prove only (i). Let $\ell_1=\mathbf{I}-\limsup x$, $\ell_2=\mathbf{I}-\limsup y$ and $\varepsilon>0$ be given. We know that both ℓ_1 and ℓ_2 are finite. Let

$$A=\{c\in\mathbb{R}\ :\ \{(n,k,l)\ :\ x_{nkl}+y_{nkl}>c\}\notin \mathbf{I}_3\}.$$

We can also assume that A is not empty. Now since

$$\left\{(n,k,l)\ :\ x_{nkl}<\ell_1+\frac{\varepsilon}{2}\right\}\cap\left\{(n,k,l)\ :\ y_{nkl}<\ell_2+\frac{\varepsilon}{2}\right\}$$
$$\subset\{(n,k,l)\ :\ x_{nkl}+y_{nkl}<\ell_1+\ell_2+\varepsilon\}$$

we have

$$\{(n,k,l)\ :\ x_{nkl}+y_{nkl}>\ell_1+\ell_2+\varepsilon\}$$
$$\subset\left\{(n,k,l)\ :\ x_{nkl}>\ell_1+\frac{\varepsilon}{2}\right\}\cup\left\{(n,k,l)\ :\ y_{nkl}>\ell_2+\frac{\varepsilon}{2}\right\}.$$

Since both sets on the right-hand side belong to $\mathbf{I}_3$ we conclude

$$\{(n,k,l)\ :\ x_{nkl}+y_{nkl}>\ell_1+\ell_2+\varepsilon\}\in \mathbf{I}_3.$$

If $c\in A,$ then $\{(n,k,l)\ :\ x_{nkl}+y_{nkl}>c\}\notin \mathbf{I}_3.$ We claim that $c\le \ell_1+\ell_2+\varepsilon.$ For, otherwise we would have

$$\{(n,k,l)\ :\ x_{nkl}+y_{nkl}>c\}\subset \{(n,k,l)\ :\ x_{nkl}+y_{nkl}>\ell_1+\ell_2+\varepsilon\}$$

which means $\{(n,k,l)\ :\ x_{nkl}+y_{nkl}>c\}\in \mathbf{I}_3,$ a contradiction. Hence $c\le \ell_1+\ell_2+\varepsilon$ and we deduce

$$\mathbf{I}_3-\limsup(x_{nkl}+y_{nkl})=\sup A\le \ell_1+\ell_2+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, this completes the proof.
We need the following definition for the subsequent theorem.

Definition 12. *Let* $\mathbf{I}_3$ *be an ideal of* $2^{\mathrm{N}\times\mathrm{N}\times\mathrm{N}}$. *A sequence* $x=(x_{nkl})$ *is said to be* $\mathbf{I}-$ *convergent to* $+\infty$ *in Pringsheim's sense* $(\text{or} -\infty)$ *if for every real number* $G>0,$ $\{(n,k,l)\ :\ x_{nkl}\le G\}\in \mathbf{I}_3$ $(\text{or } \{(n,k,l)\ :\ x_{nkl}\ge -G\}\in \mathbf{I}_3).$

Theorem 7. *If* $\mathbf{I}_3$ *is an admissible ideal and* $\mathbf{I}-\limsup x=\ell,$ *then there exists a subsequence of* (x_{nkl}) *that is* $\mathbf{I}-$ *convergent to* ℓ *in Pringsheim's sense.*

Proof. Since $\Phi\in \mathbf{I}_3$ and $\mathbf{I}_3$ is admissible, we can assume that (x_{nkl}) is a non-constant triple sequence having infinite number of distinct elements.
Case (i) : If $\ell=-\infty.$ Then from definition, $\{t\in \mathrm{R}\ :\ M_t\notin \mathbf{I}\}=\Phi.$ Hence, if $K>0,$ then $\{(n,k,l)\ :\ x_{nkl}>-2K\}\in \mathbf{I}_3.$ Since

$$\{(n,k,l)\ :\ x_{nkl}\ge -K\}\subset \{(n,k,l)\ :\ x_{nkl}>-2K\},$$

we have $\{(n,k,l)\ :\ x_{nkl}\ge -K\}\in \mathbf{I}_3$ and so $\mathbf{I}-\lim x=-\infty.$
Case (ii) : If $\ell=+\infty$ then $\{t\in \mathrm{R}\ :\ M_t\notin \mathbf{I}\}=\mathrm{R}.$ So for any $t\in \mathrm{R},$ $\{(n,k,l)\ :\ x_{nkl}>t\}\notin \mathbf{I}_3.$ Let $x_{n_1k_1l_1}$ be an arbitrary term of (x_{nkl}) and let $A_{n_1k_1l_1}=\{(n,k,l)\ :\ x_{nkl}>x_{n_1k_1l_1}+1\}.$ Since $\Phi\in \mathbf{I}_3$, $A_{n_1k_1l_1}$ is not empty and also $A_{n_1k_1l_1}\notin \mathbf{I}_3.$ We claim that there is at least one $(n,k,l)\in A_{n_1k_1l_1}$ such that $n>n_1+1, k>k_1+1,$ $l>l_1+1.$ For, otherwise

$A_{n_1k_1l_1} \subset \{(1,1,1),(2,2,2),\ldots,(n_1,n_1,n_1),(n_1+1,n_1+1,n_1+1)\},$ which is a member of $\mathbf{I}_3$ (since $\mathbf{I}_3$ is admissible) and so $A_{n_1k_1l_1} \in \mathbf{I}_3,$ a contradiction. We call this (n,k,l) as (n_2,k_2,l_2). Thus $x_{n_2k_2l_2} > x_{n_1k_1l_1} + 1.$ Proceeding in this way we obtain a subsequence $\{x_{n_ik_il_i}\}$ of (x_{nkl}) with $x_{n_ik_il_i} > x_{n_{i-1}k_{i-1}l_{i-1}} + 1$ for all i. Since for any $K > 0,$ $\{(n_i,k_i,l_i) : x_{n_ik_il_i} \le K\}$ is a finite set, it must belong to $\mathbf{I}_3,$ because $\mathbf{I}_3$ is admissible and so $\mathbf{I}_3 - \lim x_{n_{kl}} = +\infty.$

Case (iii) : If $-\infty < \ell < +\infty.$ By Theorem 1(i), $\{(n,k,l) : x_{nkl} > \ell - 1\} \notin \mathbf{I}_3$ so that $\{(n,k,l) : x_{nkl} > \ell - 1\} \neq \Phi.$ We observe that there is at least one element, say $n_1k_1l_1,$ in this set for which $x_{n_1k_1l_1} \le \ell + \frac{1}{2},$ for otherwise $\{(n,k,l) : x_{nkl} > \ell - 1\} \subset \{(n,k,l) : x_{nkl} > \ell + \frac{1}{2}\} \in \mathbf{I}_3$ which is a contradiction. Hence we have

$$\ell - 1 < x_{n_1k_1l_1} \le \ell + \frac{1}{2} < \ell + 1.$$

Next we proceed to choose an element $x_{n_2k_2l_2}$ from $(x_{nkl}),$ $n_2 > n_1$, $k_2 > k_1$ and $l_2 > l_1$ such that $x_{n_2k_2l_2} > \ell - \dfrac{1}{2},$ for otherwise $\{(n,k,l) : x_{nkl} > \ell - \frac{1}{2}\} \subset \{(1,1,1),(2,2,2),\ldots,(n_1,n_1,n_1)\}$ is a member of $\mathbf{I}_3$ which contradicts (i) of Theorem 1. Hence

$$\left\{(n,k,l) : n > n_1, k > k_1, l > l_1 \text{ and } x_{nkl} > \ell - \frac{1}{2}\right\} = E_{n_1k_1l_1}(say) \neq \Phi.$$

Now if $(n,k,l) \in E_{n_1k_1l_1}$ always implies $x_{nkl} \ge \ell + \frac{1}{2}$ then

$$E_{n_1k_1l_1} \subset \left\{(n,k,l) : x_{nkl} \ge \ell + \frac{1}{2}\right\} \subset \left\{(n,k,l) : x_{nkl} > \ell + \frac{1}{4}\right\}.$$

By (i) of Theorem 1, the right-hand set belongs to $\mathbf{I}_3$ and so $E_{n_1k_1l_1} \in \mathbf{I}_3.$ Since $\mathbf{I}_3$ is admissible, $\{(1,1,1),(2,2,2),\ldots,(n_1,n_1,n_1)\} \in \mathbf{I}_3$ and thus

$$\left\{(n,k,l) : x_{nkl} > \ell - \frac{1}{2}\right\} \subset \{(1,1,1),(2,2,2),\ldots,(n_1,n_1,n_1)\} \cup E_{n_1k_1l_1}.$$

So $\{(n,k,l) : x_{nkl} > \ell - \frac{1}{2}\} \in \mathbf{I}_3$, a contradiction to Theorem 1.

The above analysis therefore shows that there is $n_2 \circledast n_1$, $k_2 > k_1$

and $l_2 > l_1$ such that $\ell - \frac{1}{2} < x_{n_2 k_2 l_2} < \ell + \frac{1}{2}$. Proceeding in this way we obtain a subsequence $\{x_{n_i k_i l_i}\}$ of (x_{nkl}), $n_i > n_{i-1}, k_i > k_{i-1}$ and $l_i > l_{i-1}$ such that $\ell - \frac{1}{i} < x_{n_i k_i l_i} < \ell + \frac{1}{i}$ for each i. The subsequence $\{x_{n_i k_i l_i}\}$ therefore converges to ℓ in Pringsheim's sense and is thus $\mathtt{I}-$ convergent to ℓ in Pringsheim's sense by Remark 1. This proves the theorem.

The reasonings made up to now give the folowing results.

Theorem 8. *If* $\mathtt{I}-\liminf x = \ell$, *then there is a subsequence of* (x_{nkl}) *which is* $\mathtt{I}-$ *convergent to* ℓ *in Pringsheim's sense.*

Theorem 9. *Every* $\mathtt{I}-$ *bounded triple sequence* (x_{nkl}) *in Pringsheim's sense has a subsequence which is* $\mathtt{I}-$ *convergent to a finite real number in Pringsheim's sense.*

References

1. H. Fast, Sur la convergence statistique, *Colloq. Math*., 2(1951), 241-244.
2. A.R. Freedman and J.J. Sember, Densities and summability, *Pacitific J. Math.*, 95(1981), 10-11.
3. J.A. Fridy, Statistical limit points, *Proc. Amer. Math. Soc.*, 118(1993), 1187-1192.
4. J.A. Fridy and C. Orhan, Statistical limit superior and inferior, *Proc. Amer. Math. Soc.*, 125(1997), 3625-3631.
5. H.J. Hamilton, Transformations of multiple sequences, *Duke Math. J.*, 2(1936), 29-60.
6. J.L. Kelley, General Topology, Springer-Verlag, New York (1955).
7. P. Kostyrko, M. Macaj and T. Salat, $\mathtt{I}-$ Convergence, *Real Anal. Exchange, 26(2)(2000), 669-686.*
8. P. Kostyrko, M. Macaj, T. Salat and M.Sleziak, $\mathtt{I}-$ Convergence and Extremal $\mathtt{I}-$ Limit Points, *Math. Slovaca*, 55(2005), 443-464.
9. A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, *Math. Ann.*, 53(1900), 289-321.
10. G.M. Robinson, Divergent double sequences and series, *Trans. Amer. Math. Soc.*, 28(1926), 50-73.
11. H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math*., 2(1951), 73-74.
12. B.C. Tripathy, Statistically convergent double sequences, *Tamkang J. Math.*, 34:3(2003), 231-237.
13. B.K. Tripathy and B.C. Tripathy, On $\mathtt{I}-$ convergent double sequences,

Soochow J. of Math., 31(4)(2005), 549-560.
14. A. Şahiner, M. Gürdal, F.K. Düden, Triple sequences and their statistical convergence, *Selçuk J. Appl. Math*., 8(2)(2007), 49-55.