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On new version of strong Hadamard exponential function

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Abstract. It is well known that the matrix versions of the familiar real and complex exponential functions are fundamental for the study of many aspects of matrix theory and matrix group theory. In this note, we first define the *strong Hadamard product* and the *strong Hadamard exponential function* for some special block matrices, and obtain various calculus formulas for the *strong Hadamard exponential*s of some special block triangular matrices. We then give an application of this product for block-diagonal matrix with a special tridiagonal Cauchy-Toeplitz matrix.

Key words: *Matrix Group; Hadamard Product; Matrix Functions.*
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1. Introduction

For any real-valued function 5, one can define a corresponding matrix-valued function C on the space of real symmetric matrices by applying $n \geq 2$, to the eigenvalues in the spectral decomposition of 1. Matrix functions play an important role in scientific computing and engineering. Well-known examples of matrix function include $n \times n$ (the square root function of a positive semidefinite matrix), and $e^{?N}$ (the exponential function of a square matrix) [8]. Let A be an $n \times n$ real or complex matrix. By [2] the exponential of A , denoted by e^A or $\exp(A)$, is the $n \times n$ matrix given by the power series

$$e^A = \sum_{n \geq 0} \frac{1}{n!} A^n.$$

Löwner [13] first introduced the notion of matrix monotone functions in 1934. Two years later, Löwner's student Kraus [11] extended his work to matrix convex functions. The standard matrix analysis books of Bhata [5] and Horn and

Johnson [9] contain more historical notes and related literature on this class of matrix functions.
A notion that is useful in the study of matrix equations and other applications, and is of interest in its own right, is the *Kronecker product, direct product, or tensor product of matrices.* In [20], the Kronecker product of two matrices A and B of sizes $m \times n$ and $s \times t$, respectively, in over a field R or C is defined to be the following $(ms) \times (nt)$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}.$$

The Hadamard product, or the Schur product, of two matrices A and B of the same size is defined to be the entrywise product

$$A \circ B = (a_{ij}b_{ij}) \,,$$

(see [20]).
It is well known that if A and B are the $n \times n$ matrices, then the Hadamard product $A \circ B$ is a principal submatrix of the Kronecker product $A \otimes B$ lying on intersections of rows and columns 1, $n + 2$, $2n + 3$, ..., n^2 (see [20]).
The Hadamard product arises in a wide variety of ways. Several examples, such as trigonometric moments of convolutions of periodic functions, products of integral equation kernels, the weak minimum principle in partial differential equations, and characteristic functions in probability theory are discussed in section (7.5) of [9]. The Hadamard unit matrix U is such a matrix whose all entries are 1 (the size of U being understood). The matrix A called *Hadamard invertible* if all its entries are non-zero. Then $A^{\circ -1} = \left(a_{ij}^{-1}\right)$ is then called the Hadamard inverse of A.
A class of Hermitian matrices with a special positivity property arises naturally in many applications. Hermitian (and, in particular, real symmetric) matrices with this positivitiy property also provide one generalization to matrices of the notion of a positive number. This observation often provides insight into the properties and applications of positive semidefinite matrices. An n-by-n Hermitian matrix A is said to be *positive semidefinite* if

$$x^*Ax \geq 0 \text{ for all nonzero } x \in \mathbb{C}^n.$$

For a short survey of facts about real positive definite matrices see [10].
Bapat [3], [4] showed that if A is symmetric, while it has all positive entries and just one positive eigenvalue, then its Hadamard inverse $A^{\circ -1}$ is positive semidefinite. If A is a distance matrix, then the Hadamard square root of A has just one positive eigenvalue, and is invertible. This was proved most recently by Auer [1] , and it had been proved by Schoenberg [15].
For A in $\mathrm{C}^{m \times n}$, the power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} A^{\circ k},$$

converges normally (which means that the series of norms is convergent), since, for any matrix norm, it is well known that [14]

$$\|A \circ A\| \leq \|A\|^2 ,$$

and we have

$$\sum_{k=0}^{\infty} \left\| \frac{1}{k!} A^{\circ k} \right\| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|A\|^k = e^{\|A\|}.$$

Since $\mathbb{C}^{m\times n}$ is complete, the series is convergent, and the estimation above shows that it converges uniformly on every compact set. Its sum, denoted by $e^{\circ A}$, thus defines a continuous map $\exp : \mathbb{C}^{m\times n} \to \mathbb{C}^{m\times n}$, called the *exponential*. When $A \in \mathbb{R}^{m\times n}$, we have $e^{\circ A} \in \mathbb{R}^{m\times n}$.
Reams [18] proved that if $A \in \mathbb{R}^{n\times n}$ is positive semidefinite, then the Hadamard exponential

$$e^{\circ A} = U + A + \frac{A^{\circ 2}}{2!} + \frac{A^{\circ 3}}{3!} + ... + \frac{A^{\circ n}}{n!} + ...,$$

where U is the $n \times n$ Hadamard unit matrix, is positive semidefinite.
Let M be a square matrix of order tm partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{t1} & M_{t2} & \cdots & M_{tt} \end{bmatrix},$$

where M_{ij} is a square matrix of order m $(i, j = 1, 2, ..., t)$ and N be a square matrix of order tn partitioned as

$$N = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1t} \\ N_{21} & N_{22} & \cdots & N_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ N_{t1} & N_{t2} & \cdots & N_{tt} \end{bmatrix},$$

where N_{ij} is a square matrix of order n $(i, j = 1, 2, ..., t)$. In [16] Seberry and Zhang defined the operation $\circledast$ as the follows:

$$M \circledast N = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1t} \\ L_{21} & L_{22} & \cdots & L_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ L_{t1} & L_{t2} & \cdots & L_{tt} \end{bmatrix},$$

where L_{ij} is a square matrix of order mn and

$$L_{ij} = M_{i1} \otimes N_{1j} + M_{i2} \otimes N_{2j} + ... + M_{it} \otimes N_{tj},$$

where $\otimes$ is Kronecker product, $i, j = 1, 2, ..., t$. This product is called as the strong Kronecker product.

The strong Kronecker product is developed in [16] and supported the analysis of certain orthogonal matrix multiplication problems. The strong Kronecker product is considered a powerful matrix multiplication tool for Hadamard and other orthogonal matrices from combinatorial theory [12].

In the second section of this paper we introduce the *strong Hadamard product* and the *strong Hadamard exponential* of some special block matrices, and obtain various calculus formulas for the strong exponentials of some special block triangular matrices. In the third section of this paper we give an application of this product for block-diagonal matrix with a special tridiagonal Cauchy-Toeplitz matrix.

2. The new version of the strong Hadamard exponential function

Definition 1. *Let D and E be a real matrices partitioned as*

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

where D_{ij} and $E_{ij}, 1 \leq i,\, j \leq 2$, are same size matrices. The strong Hadamard product $D \odot E$ of the matrices D and E is defined as

$$D \odot E = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$$

where $F_{ij} = \sum\limits_{k=1}^{2} D_{ik} \circ E_{kj}$.

Definition 2. *Let H be a real matrix partitioned as*

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

where H_{ij}, $1 \leq i, j \leq 2$, are $m \times n$ matrices. The strong exponential of H, denoted by $e^{\odot H}$ or $exp^{\odot}(H)$, is the $2m \times 2n$ matrix given by the power series

$$e^{\odot H} = \sum_{n \geq 0} \frac{1}{n!} H^{\odot n} = U + \frac{H}{1!} + \frac{H^{\odot 2}}{2!} + ... + \frac{H^{\odot n}}{n!} + ..., \tag{2.1}$$

where U is the $2m \times 2n$ Hadamard unit matrix.

In this study, for the matrix

$$M = \begin{pmatrix} A & B \\ 0 & U \end{pmatrix}, \tag{2.2}$$

we also define

$$M^{\odot(0)} = \begin{pmatrix} U & U \\ 0 & U \end{pmatrix},$$

where $A = [a_{ij}], B, U, 0 \in \mathbb{R}^{m\times n}$ and U is Hadamard unit matrix.

Theorem 3. *Let M be matrix*given in (2.2). For $a, b \in \mathbb{R}$,

$$e^{\odot(a+b)M} = \begin{pmatrix} e^{\circ[(a+b)A]} & U + \left\{(a+b)U + \left[\sum_{n\geq 2}\frac{1}{n!}\sum_{k=0}^{n-1}(a+b)^{k+1}a_{ij}^k\right]\right\}\circ B \\ 0 & e^{\circ[(a+b)U]} \end{pmatrix},$$

when $a_{ij} \neq \frac{1}{a+b}$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

Proof : Let
$\Omega = e^{\odot(a+b)M} - \begin{pmatrix} U & U \\ 0 & U \end{pmatrix} - \begin{pmatrix} (a+b)A & (a+b)B \\ 0 & (a+b)U \end{pmatrix}$. By expanding the series Ω we get

$$\begin{aligned}
\Omega &= \sum_{n\geq 2}\frac{1}{n!}\begin{pmatrix} (a+b)^n A^{\circ n} & \left[\sum_{k=0}^{n-1}(a+b)^{k+1}a_{ij}^k\right]\circ B \\ 0 & (a+b)^n U \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n\geq 2}\frac{1}{n!}(a+b)^n A^{\circ n} & \left[\sum_{n\geq 2}\frac{1}{n!}\sum_{k=0}^{n-1}(a+b)^{k+1}a_{ij}^k\right]\circ B \\ 0 & \sum_{n\geq 2}\frac{1}{n!}(a+b)^n U \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n\geq 2}\frac{1}{n!}\left(\sum_{r=0}^{n}\begin{pmatrix} n \\ r\end{pmatrix}a^r b^{n-r}\right)A^{\circ n} & \left[\sum_{n\geq 2}\frac{1}{n!}\sum_{k=0}^{n-1}(a+b)^{k+1}a_{ij}^k\right]\circ B \\ 0 & \sum_{n\geq 2}\frac{1}{n!}\left(\sum_{r=0}^{n}\begin{pmatrix} n \\ r\end{pmatrix}a^r b^{n-r}\right)U \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n\geq 2, s\geq 2}\frac{a^r b^s}{r!s!}A^{\circ(r+s)} & \left[\sum_{n\geq 2}\frac{1}{n!}\sum_{k=0}^{n-1}(a+b)^{k+1}a_{ij}^k\right]\circ B \\ 0 & \sum_{n\geq 2, s\geq 2}\frac{a^r b^s}{r!s!}U \end{pmatrix} \\
&= \begin{pmatrix} \left(\sum_{n\geq 2}\frac{a^r}{r!}A^{\circ r}\right)\circ\left(\sum_{s\geq 2}\frac{a^s}{s!}A^{\circ s}\right) & \left[\sum_{n\geq 2}\frac{1}{n!}\sum_{k=0}^{n-1}(a+b)^{k+1}a_{ij}^k\right]\circ B \\ 0 & \left(\sum_{n\geq 2}\frac{a^r}{r!}U\right)\circ\left(\sum_{s\geq 2}\frac{a^s}{s!}U\right) \end{pmatrix}.
\end{aligned}$$

Consequently, we get

$$
\begin{aligned}
e^{\odot(a+b)M} &= \begin{pmatrix} e^{\circ(aA)}\circ e^{\circ(bA)} & U+(a+b)B+\left[\sum\limits_{n\geq 2}\frac{1}{n!}\sum\limits_{k=0}^{n-1}(a+b)^{k+1}a_{ij}^{k}\right]\circ B \\ 0 & e^{\circ(aU)}\circ e^{\circ(bU)} \end{pmatrix} \\
&= \begin{pmatrix} e^{\circ[(a+b)A]} & U+(a+b)B+\left[\sum\limits_{n\geq 2}\frac{1}{n!}\sum\limits_{k=0}^{n-1}(a+b)^{k+1}a_{ij}^{k}\right]\circ B \\ 0 & e^{\circ[(a+b)U]} \end{pmatrix},
\end{aligned}
$$

which proves Theorem 3.

Corollary 4. *Let M be matrix given in* (2.2)*. Then*

$$
e^{\odot M} = \begin{pmatrix} e^{\circ A} & U+\left\{U+\left[\sum\limits_{n\geq 2}\frac{1}{n!}\sum\limits_{k=0}^{n-1}a_{ij}^{k}\right]\right\}\circ B \\ 0 & e^{\circ U} \end{pmatrix},
$$

when $a_{ij}\neq 1$, $i=1,2,...,m$, $j=1,2,...,n$.

Proof : The proof is similar to the proof of Theorem 3, and we only show an outline of it. Let

$$
\Phi = e^{\odot M} - \begin{pmatrix} U & U \\ 0 & U \end{pmatrix} - \begin{pmatrix} A & B \\ 0 & U \end{pmatrix}.
$$

By expanding the series Φ and performing a sequence of manipulations we obtain

$$
\begin{aligned}
\Phi &= \sum_{n\geq 2}\frac{1}{n!}M^{\odot n} \\
&= \sum_{n\geq 2}\frac{1}{n!}\begin{pmatrix} A^{\circ n} & \left(A^{\circ(n-1)}+A^{\circ(n-2)}+...+U\right)\circ B \\ 0 & U \end{pmatrix}.
\end{aligned}
$$

Hence, we have

$$
e^{\odot M} = \begin{pmatrix} e^{\circ A} & U+B+\left[\sum\limits_{n\geq 2}\frac{1}{n!}\sum\limits_{k=0}^{n-1}a_{ij}^{k}\right]\circ B \\ 0 & e^{\circ U} \end{pmatrix}.
$$

This completes the proof.

Corollary 5. *Let M be matrix given in* (2.2)*. Then*

$$
e^{\odot(-M)} = \begin{pmatrix} \left(e^{\circ A}\right)^{\circ -1} & U-\left\{U+\left[\sum\limits_{n\geq 2}\frac{1}{n!}\sum\limits_{k=0}^{n-1}(-a_{ij})^{k}\right]\right\}\circ B \\ 0 & \left(e^{\circ U}\right)^{\circ -1} \end{pmatrix},
$$

when $a_{ij}\neq -1$, $i=1,2,...,m$, $j=1,2,...,n$.

Proof : Corollary 5 follows from Theorem 3.

Theorem 6. *Let*

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ 0 & U \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} A_2 & B_2 \\ 0 & U \end{pmatrix},$$

where $A_1 = \left[a_{ij}^{(1)}\right], A_2 = \left[a_{ij}^{(2)}\right], B_1, B_2, U, 0 \in \mathbb{R}^{m\times n}$ and U is Hadamard unit matrix. Then

$$e^{\odot(M_1+M_2)} = \begin{pmatrix} e^{\circ(A_1+A_2)} & U + \left\{ U + \left[\sum_{n\geq 2} \frac{1}{n!} \sum_{k=0}^{n-1} 2^{n-1-k} \left(a_{ij}^{(1)} + a_{ij}^{(2)}\right)^k \right] \right\} \circ (B_1 + B_2) \\ 0 & e^{\circ(2U)} \end{pmatrix},$$

when $a_{ij}^{(1)} + a_{ij}^{(2)} \neq 1$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

Proof : Let

$\Psi = e^{\odot(M_1+M_2)} - \begin{pmatrix} U & U \\ 0 & U \end{pmatrix} - \begin{pmatrix} A_1 + A_2 & B_1 + B_2 \\ 0 & U \end{pmatrix}$. From (2.1), we get

$$\begin{aligned}
\Psi &= \sum_{n\geq 2} \frac{1}{n!} \begin{pmatrix} (A_1 + A_2)^{\circ n} & \left[\sum_{k=0}^{n-1} 2^{n-1-k} \left(a_{ij}^{(1)} + a_{ij}^{(2)}\right)^k\right] \circ (B_1 + B_2) \\ 0 & (U + U)^{\circ n} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} A_1^{\circ r} A_2^{\circ(n-r)} \right) & * \\ 0 & \sum_{n\geq 0} \frac{1}{n!} \left(\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} U^{\circ r} U^{\circ(n-r)} \right) \end{pmatrix} \\
&= \begin{pmatrix} \left(\sum_{n\geq 0} \frac{1}{r!} A_1^{\circ r} \right) \circ \left(\sum_{s\geq 0} \frac{1}{s!} A_2^{\circ s} \right) & * \\ 0 & \left(\sum_{n\geq 0} \frac{1}{r!} U \right) \circ \left(\sum_{s\geq 0} \frac{1}{s!} U \right) \end{pmatrix} \\
&= \begin{pmatrix} e^{\circ(A_1)} \circ e^{\circ(A_2)} & * \\ 0 & e^{\circ(U)} \circ e^{\circ(U)} \end{pmatrix} \\
&= \begin{pmatrix} e^{\circ(A_1+A_2)} & * \\ 0 & e^{\circ(2U)} \end{pmatrix},
\end{aligned}$$

where the matrix $*$ is $\left[\sum_{n\geq 2} \frac{1}{n!} \sum_{k=0}^{n-1} 2^{n-1-k} \left(a_{ij}^{(1)} + a_{ij}^{(2)}\right)^k \right] \circ (B_1 + B_2)$. From this, the proof is completed.

For Theorem 8, we need to give the following theorem:

Theorem 7. *Let M be a square complex matrix partitioned as*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, and D are $m\times m$, $m\times n$, $n\times m$ and $n\times n$ matrices, respectively. Then,

$$\det M=\begin{cases}\det A\det\left(D-CA^{-1}B\right), & \text{if } A \text{ is invertible}\\ \det\left(AD-CB\right), & \text{if } AC=CA\end{cases}.$$

Theorem 8. Let

$$M=\begin{pmatrix} A & B\\ 0 & I\end{pmatrix},$$

where $A=[a_{ij}],B,I,0\in\mathbb{R}^{n\times n}$ and I is the ordinary identity matrix. If A is positive semidefinite matrix, then

$$\det\left(e^{\odot M}\right)\leq e^{n+trA},$$

where trA is trace of the matrix A.
Proof. From (2.1) we write

$$e^{\odot M}=\begin{pmatrix} e^{\circ A} & *\\ 0 & e^{\circ I}\end{pmatrix},$$

where the matrix $*$ is $U+B+\left[A^{\circ(n-1)}+\left(A^{\circ(n-2)}+A^{\circ(n-3)}...+A^{\circ 0}\right)\circ I\right]\circ B$, $n\geq 2$, with $A^{\circ 0}=U$. Here, U is the Hadamard unit matrix. Since A is positive semidefinite matrix, evidently, $e^{\circ A}=U+A+\frac{1}{2!}A^{\circ 2}+...$ is positive semidefinite (see [18]). Similarly, $e^{\circ I}$ is positive semidefinite. From the definition of the strong Hadamard exponential function and Theorem 7, it is written immediately that

$$\det\left(e^{\odot M}\right)=\det e^{\circ A}\det e^{\circ I}.$$

Let $D=diag\left(e^{-\frac{1}{2}a_{11}},e^{-\frac{1}{2}a_{22}},...,e^{-\frac{1}{2}a_{nn}}\right)$. Let Λ be a matrix such that $\Lambda=De^{\circ A}D$. Then Λ is a positive semidefinite matrix with diagonal entries all equal to 1. From the arithmetic-geometric mean inequality, it follows

$$n=tr\Lambda=\sum\lambda_i\left(\Lambda\right)\geq n\left[\prod_{i=1}^{n}\lambda_i\left(\Lambda\right)\right]^{\frac{1}{n}}=n\left(\det\Lambda\right)^{\frac{1}{n}},$$

(also see [20], p.176). This implies $\det\left(\Lambda\right)\leq 1$. Thus,

$$\det e^{\circ A}=\det\left(D^{-1}\Lambda D^{-1}\right)=\prod_{i=1}^{n}e^{a_{ii}}\det\Lambda\leq\prod_{i=1}^{n}e^{a_{ii}}.$$

Similarly, we obtain

$$\det e^{\circ I}\leq e^{n}.$$

Consequently,

$$\det\left(e^{\odot M}\right)\leq e^{n+trA},$$

where trA is trace of the matrix A.

3. More on the strong Hadamard exponential function

A Cauchy-Toeplitz matrix is a matrix that is both a Cauchy matrix (i.e. $\left[\frac{1}{x_i-y_j}\right]_{i,j=1}^{n}, x_i \neq y_j$) and a Toeplitz matrix (i.e. $(z_{i-j})_{i,j=1}^{n}$) such that

$$T_n\left(g,h\right) = \left[\frac{1}{g+(i-j)h}\right]_{i,j=1}^{n},$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not integer.
Parter [17] has given a qualitative explanation of phenomena which is that most of singular values (first 20) of the Cauchy-Toeplitz matrix $T_n\left(1/2,1\right)$ were equal to $\pi - \varepsilon$ with ε very small.. Tyrtyshnikov [19] has shown that minimal singular values of the Cauchy-Toeplitz matrix $T_n\left(1/2,1\right)$ converge to zero with increasing n.
The interest of the study of tridiagonal Toeplitz matrices appears to be very important not only from a theoretical point of view (in linear algebra or numerical analysis), but also in applications. For instance, it is useful in the study of sound propagation problems [6].
A positive tridiagonal Cauchy-Toeplitz matrix is the $n \times n$ tridiagonal matrix with entries $m_{k,k} = 1/g$, $1 \leq k \leq n$, and $m_{k-1,k} = m_{k,k-1} = 1/\left(g+h\right)$, $2 \leq k \leq n$, that is,

$$T_n\left(g,h\right) = \begin{pmatrix} 1/g & 1/\left(g+h\right) & & \\ 1/\left(g+h\right) & 1/g & \ddots & \\ & \ddots & \ddots & 1/\left(g+h\right) \\ & & 1/\left(g+h\right) & 1/g \end{pmatrix}, \tag{3.1}$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not integer.
Let T_n be matrix given in (3.1). In this section, we will use the notation $T = T_n\left(1/2,1\right)$.

Theorem 9. Let $N = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$. Then $e^{\odot N}$ and $e^{\odot(-N)}$ are positive semidefinite.

Proof. From (2.1), we obtain

$$e^{\odot N} = \begin{pmatrix} e^{\circ T} & 0 \\ 0 & e^{\circ T} \end{pmatrix}$$

and

$$e^{\odot(-N)} = \begin{pmatrix} \left(e^{\circ T}\right)^{\circ -1} & 0 \\ 0 & \left(e^{\circ T}\right)^{\circ -1} \end{pmatrix}.$$

We now introduce a sequence of matrices $\{G_n, n = 1, 2, ...\}$, where G_n is the $n \times n$ tridiagonal matrix with entries $g_{k,k} = 0$, $1 \leq k \leq n$, and $g_{k-1,k} = g_{k,k-1} = 1$, $2 \leq k \leq n$. That is,

$$G_n = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

Then, note that $T = 2I_n + \frac{2}{3}G_n$. Let λ_k, $k = 1, 2, .., n$, be the eigenvalues of the matrix G_n (with associated eigenvectors x_k). Then, for each j,

$$\begin{aligned} Tx_j &= \left[2I_n + \frac{2}{3}G_n\right] x_j \\ &= \left[2 + \frac{2}{3}\lambda_k\right] x_j. \end{aligned}$$

Therefore, $\theta_k = 2 + \frac{2}{3}\lambda_k$, $k = 1, 2, .., n$, are the eigenvalues of T. Since it was determined the λ_k's as

$$\lambda_k = -2\cos\frac{\pi k}{n+1}, \quad k = 1, 2, ..., n$$

in [7], evidently, $e^{\circ T} = U + T + \frac{1}{2!}T^{\circ 2} + ...$ is positive semidefinite, where U is the Hadamard unit matrix. Therefore, we also get that the matrix $e^{\odot N}$ is positive semidefinite. Also, since $e^{\circ T}$ have positive entries and just one positive eigenvalue, the Hadamard inverse $\left(e^{\circ T}\right)^{\circ -1}$ is positive semidefinite. Thus, $e^{\odot(-N)}$ is positive semidefinite.

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