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On The Recursive Sequence $x_{n+1} = \frac{x_{n-11}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}$

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Summary. In this paper a solution of the following difference equation was investigated

$$x_{n+1} = \frac{x_{n-11}}{1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}},\ n = 0, 1, 2, ...$$

where $x_{-11}, x_{-10}, ..., x_{-2}, x_{-1}, x_0 \in (0, \infty)$

Key words: Difference equation, period twelve solution.

1. Introduction

Recently there has been a lot of interest in studying the periodic nature of nonlinear difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations see, for examples [1,2,4,5].

In [3] the following problem was posed. There is a solution of the following difference equation

$$x_{n+1} = \frac{\beta x_{n-1}}{\beta + x_n} \text{ for } n = 0, 1, 2, ...$$

where $x_{-1}, x_0 \in (0, \infty)$, $\beta > 0$ such that $x_n \to 0$ as $n \to \infty$.

In [6] Stevic assumed that $\beta = 1$ and solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n} \text{ for } n = 0, 1, 2, ...$$

where $x_{-1}, x_0 \in (0, \infty)$. Also, this result was generalized to the equation of the following form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)} \text{ for } n = 0, 1, 2, ...$$

where $x_{-1}, x_0 \in (0, \infty)$.
In this paper we investigated the following nonlinear difference equation

$$(1) \qquad x_{n+1} = \frac{x_{n-11}}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}} \text{ for } n = 0, 1, 2, ...$$

where $x_{-11}, x_{-10}, ..., x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

2. Main Result

Teorem 1 Consider the difference equation (1). Then the following statements are true.
a) *The sequences* $(x_{12n-11}), (x_{12n-10}), (x_{12n-9}), (x_{12n-8}), (x_{12n-7}), (x_{12n-6}),$
$(x_{12n-5}), (x_{12n-4}), (x_{12n-3}), (x_{12n-2}), (x_{12n-1})$ *and* (x_{12n}) *are decreasing and there exist*
$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12} \geqslant 0$ *such that*

$$\lim_{n\to\infty} x_{12n-11} = a_1, \lim_{n\to\infty} x_{12n-10} = a_2, \lim_{n\to\infty} x_{12n-9} = a_3, \lim_{n\to\infty} x_{12n-8} = a_4,$$

$$\lim_{n\to\infty} x_{12n-7} = a_5, \lim_{n\to\infty} x_{12n-6} = a_6, \lim_{n\to\infty} x_{12n-5} = a_7, \lim_{n\to\infty} x_{12n-4} = a_8,$$

$$\lim_{n\to\infty} x_{12n-3} = a_9, \lim_{n\to\infty} x_{12n-2} = a_{10}, \lim_{n\to\infty} x_{12n-1} = a_{11}, \lim_{n\to\infty} x_{12n} = a_{12}.$$

b) $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12},$
$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, ...)$ *is a solution of equation (1) of period twelve.*
c) $a_1.a_3.a_5.a_7.a_9.a_{11} = 0$, $a_2.a_4.a_6.a_8.a_{10}.a_{12} = 0$.
d) *If there exists* $n_0 \in N$ *such that*
$x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9} \geq x_{n+1}x_{n-1}x_{n-3}x_{n-5}x_{n-7}$
for all $n \geqslant n_0$*, then*

$$\lim_{n\to\infty} x_n = 0.$$

e) *The following formulas*

$$x_{12n+1} = x_{-11}(1 - \frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{n} \prod_{i=1}^{6j} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+2}=x_{-10}(1-\frac{x_0x_{-2}x_{-4}x_{-6}x_{-8}}{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+3}=x_{-9}(1-\frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+4}=x_{-8}(1-\frac{x_0x_{-2}x_{-4}x_{-6}x_{-10}}{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+5}=x_{-7}(1-\frac{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+6}=x_{-6}(1-\frac{x_0x_{-2}x_{-4}x_{-8}x_{-10}}{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+7}=x_{-5}(1-\frac{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+8}=x_{-4}(1-\frac{x_0x_{-2}x_{-6}x_{-8}x_{-10}}{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+9}=x_{-3}(1-\frac{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+10}=x_{-2}(1-\frac{x_0x_{-4}x_{-6}x_{-8}x_{-10}}{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+11}=x_{-1}(1-\frac{x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+12}=x_0\left(1-\frac{x_{-2}x_{-4}x_{-6}x_{-8}x_{-10}}{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}\right),$$

hold.

f) *If* $x_{12n+1}\to a_1\neq 0$, $x_{12n+3}\to a_3\neq 0$, $x_{12n+5}\to a_5\neq 0$, $x_{12n+7}\to a_7\neq 0$ *and* $x_{12n+9}\to a_9\neq 0$ *then* $x_{12n+11}\to 0$ *as* $n\to\infty$.
If $x_{12n+2}\to a_2\neq 0$, $x_{12n+4}\to a_4\neq 0$, $x_{12n+6}\to a_6\neq 0$, $x_{12n+8}\to a_8\neq 0$ *and* $x_{12n+10}\to a_{10}\neq 0$ *then* $x_{12n+12}\to 0$ *as* $n\to\infty$.

Proof 1 a) Firstly, we consider equation (1). From this equation, we obtain

$$x_{n+1}(1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9})=x_{n-11}.$$

If $x_{n-1},x_{n-3},x_{n-5},x_{n-7},x_{n-9}\in(0,+\infty)$, then $(1+x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9})\in(1,+\infty)$. Since $x_{n+1}<x_{n-11}$, $n\in N$, we obtain that there exist
$\lim_{n\to\infty}x_{12n-11}=a_1$, $\lim_{n\to\infty}x_{12n-10}=a_2$, $\lim_{n\to\infty}x_{12n-9}=a_3$, $\lim_{n\to\infty}x_{12n-8}=a_4$,
$\lim_{n\to\infty}x_{12n-7}=a_5$, $\lim_{n\to\infty}x_{12n-6}=a_6$, $\lim_{n\to\infty}x_{12n-5}=a_7$, $\lim_{n\to\infty}x_{12n-4}=a_8$,
$\lim_{n\to\infty}x_{12n-3}=a_9$, $\lim_{n\to\infty}x_{12n-2}=a_{10}$, $\lim_{n\to\infty}x_{12n-1}=a_{11}$, $\lim_{n\to\infty}x_{12n}=a_{12}$.

b) $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},$
$a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},...)$ *is a solution of equation (1) of period twelve.*

c) In view of equation (1), we obtain

$$x_{12n+1}=\frac{x_{12n-11}}{1+x_{12n-1}x_{12n-3}x_{12n-5}x_{12n-7}x_{12n-9}}.$$

Take the limits on both sides of the above equality

$$a_1=\frac{a_1}{1+a_{11}.a_9.a_7.a_5.a_3}\Rightarrow a_1+a_{11}.a_9.a_7.a_5.a_3.a_1=a_1\Rightarrow a_1.a_3.a_5.a_7.a_9.a_{11}=0.$$

Also, we obtain

$$x_{12n+2}=\frac{x_{12n-10}}{1+x_{12n}x_{12n-2}x_{12n-4}x_{12n-6}x_{12n-8}}.$$

Take the limits on both sides of the above equality

$$a_2=\frac{a_2}{1+a_{12}.a_{10}.a_8.a_6.a_4}\Rightarrow a_2+a_{12}.a_{10}.a_8.a_6.a_4.a_2=a_2\Rightarrow a_2.a_4.a_6.a_8.a_{10}.a_{12}=0.$$

d) If there exists $n_0\in N$ such that
$x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}\geq x_{n+1}x_{n-1}x_{n-3}x_{n-5}x_{n-7}$

for all $n \geqslant n_0$, then $a_1 \leq a_3 \leq a_5 \leq a_7 \leq a_9 \leq a_{11} \leq a_1$ and $a_2 \leq a_4 \leq a_6 \leq a_8 \leq a_{10} \leq a_{12} \leq a_2$.
Since $a_1.a_3.a_5.a_7.a_9.a_{11} = 0$ and $a_2.a_4.a_6.a_8.a_{10}.a_{12} = 0$ we obtain the result.
e) Subracting x_{n-11} from the left and right-hand sides of equation (1) we obtain

$$x_{n+1} - x_{n-11} = \frac{1}{1 + x_{n-1}x_{n-3}x_{n-5}x_{n-7}x_{n-9}}(x_{n-1} - x_{n-13})$$

and the following formula

(2)

$$for\ n \geqslant 2 \begin{cases} x_{2n-3} - x_{2n-15} = (x_1 - x_{-11}) \prod_{i=1}^{n-2} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} \\ x_{2n-2} - x_{2n-14} = (x_2 - x_{-10}) \prod_{i=1}^{n-2} \frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} \end{cases}$$

holds. Replacing n by $6j$ in (2) and summing from $j = 0$ to $j = n$, we obtain

(3)

$$x_{12n+1} - x_{-11} = (x_1 - x_{-11}) \sum_{j=0}^{n} \prod_{i=1}^{6j} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} \quad (n = 0, 1, 2, ...)$$

and

(4)

$$x_{12n+2} - x_{-10} = (x_2 - x_{-10}) \sum_{j=0}^{n} \prod_{i=1}^{6j} \frac{1}{1 + x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} \quad (n = 0, 1, 2, ...).$$

Also, replacing n by $6j + 1$ in (2) and summing from $j = 0$ to $j = n$, we obtain

(5)

$$x_{12n+3} - x_{-9} = (x_1 - x_{-11}) \sum_{j=0}^{n} \prod_{i=1}^{6j+1} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} \quad (n = 0, 1, 2, ...)$$

and
(6)

$$x_{12n+4} - x_{-8} = (x_2 - x_{-10}) \sum_{j=0}^{n} \prod_{i=1}^{6j+1} \frac{1}{1 + x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} \quad (n = 0, 1, 2, ...).$$

Also, replacing n by $6j + 2$ in (2) and summing from $j = 0$ to $j = n$, we obtain

(7)

$$x_{12n+5} - x_{-7} = (x_1 - x_{-11}) \sum_{j=0}^{n} \prod_{i=1}^{6j+2} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} \quad (n = 0, 1, 2, ...)$$

and

(8)

$$x_{12n+6}-x_{-6} = (x_2-x_{-10})\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} \quad (n=0,1,2,...).$$

Also, replacing n by $6j+3$ in (2) and summing from $j=0$ to $j=n$, we obtain

(9)

$$x_{12n+7}-x_{-5} = (x_1-x_{-11})\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} \quad (n=0,1,2,...)$$

and

(10)

$$x_{12n+8}-x_{-4} = (x_2-x_{-10})\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} \quad (n=0,1,2,...).$$

Also, replacing n by $6j+4$ in (2) and summing from $j=0$ to $j=n$, we obtain

(11)

$$x_{12n+9}-x_{-3} = (x_1-x_{-11})\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} \quad (n=0,1,2,...)$$

and

(12)

$$x_{12n+10}-x_{-2} = (x_2-x_{-10})\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} \quad (n=0,1,2,...).$$

Also, replacing n by $6j+5$ in (2) and summing from $j=0$ to $j=n$, we obtain

(13)

$$x_{12n+11}-x_{-1} = (x_1-x_{-11})\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} \quad (n=0,1,2,...)$$

and

(14)

$$x_{12n+12}-x_{0} = (x_2-x_{-10})\sum_{j=0}^{n}\prod_{i=1}^{6j+5}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} \quad (n=0,1,2,...).$$

Therefore, we obtain following formulas;

$$x_{12n+1}=x_{-11}(1-\frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+2}=x_{-10}(1-\frac{x_{0}x_{-2}x_{-4}x_{-6}x_{-8}}{1+x_{0}x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+3}=x_{-9}(1-\frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+4}=x_{-8}(1-\frac{x_{0}x_{-2}x_{-4}x_{-6}x_{-10}}{1+x_{0}x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+1}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+5}=x_{-7}(1-\frac{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+6}=x_{-6}(1-\frac{x_{0}x_{-2}x_{-4}x_{-8}x_{-10}}{1+x_{0}x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+7}=x_{-5}(1-\frac{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+8}=x_{-4}(1-\frac{x_{0}x_{-2}x_{-6}x_{-8}x_{-10}}{1+x_{0}x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+9}=x_{-3}(1-\frac{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+10}=x_{-2}(1-\frac{x_{0}x_{-4}x_{-6}x_{-8}x_{-10}}{1+x_{0}x_{-2}x_{-4}x_{-6}x_{-8}}\sum_{j=0}^{n}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}),$$

$$x_{12n+11} = x_{-1}(1 - \frac{x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{n} \prod_{i=1}^{6j+5} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}),$$

$$x_{12n+12} = x_0(1 - \frac{x_{-2}x_{-4}x_{-6}x_{-8}x_{-10}}{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}} \sum_{j=0}^{n} \prod_{i=1}^{6j+5} \frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}).$$

f) Suppose that $a_1 = a_3 = a_5 = a_7 = a_9 = a_{11} = 0$. By (e), we have

$$\lim_{n\to\infty} x_{12n+1} = \lim_{n\to\infty} x_{-11}(1 - \frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{n} \prod_{i=1}^{6j} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}})$$

$$a_1 = x_{-11}(1 - \frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{\infty} \prod_{i=1}^{6j} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}})$$

(15)

$$a_1 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} = \sum_{j=0}^{\infty} \prod_{i=1}^{6j} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}.$$

and,

$$a_3 = x_{-9}(1 - \frac{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{\infty} \prod_{i=1}^{6j+1} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}})$$

(16)

$$a_3 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}} = \sum_{j=0}^{\infty} \prod_{i=1}^{6j+1} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}.$$

Similarly, we write

$$a_5 = x_{-7}(1 - \frac{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}}{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{\infty} \prod_{i=1}^{6j+2} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}})$$

(17)

$$a_5 = 0 \Rightarrow \frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}} = \sum_{j=0}^{\infty} \prod_{i=1}^{6j+2} \frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}.$$

Then,

$$a_7 = x_{-5}(1 - \frac{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}}{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{\infty} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}})$$

(18)

$$a_7 = 0 \Rightarrow \frac{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{\infty} \prod_{i=1}^{6j+3} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}.$$

In the same manner, we obrain

$$a_9 = x_{-3}(1 - \frac{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}}{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{\infty} \prod_{i=1}^{6j+4} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}})$$

(19)

$$a_9 = 0 \Rightarrow \frac{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{\infty} \prod_{i=1}^{6j+4} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}.$$

Finally,

$$a_{11} = x_{-1}(1 - \frac{x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}}{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} \sum_{j=0}^{\infty} \prod_{i=1}^{6j+5} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}})$$

(20)

$$a_{11} = 0 \Rightarrow \frac{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{\infty} \prod_{i=1}^{6j+5} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}.$$

From equation (15) and (16), we obtain

$$\begin{aligned} \frac{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}} &= \sum_{j=0}^{\infty} \prod_{i=1}^{6j} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} > \quad (21) \\ \frac{1 + x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}} &= \sum_{j=0}^{\infty} \prod_{i=1}^{6j+1} \frac{1}{1 + x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} \end{aligned}$$

thus, $x_{-11} > x_{-9}$.
From equation (16) and (17), we obtain

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-5}x_{-7}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} > \quad (22)$$

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}$$

thus, $x_{-9} > x_{-7}$.
From equation (17) and (18), we obtain

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-5}x_{-9}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} > \quad (23)$$

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}$$

thus, $x_{-7} > x_{-5}$.
From equation (18) and (19), we obtain

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-3}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} > \quad (24)$$

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}$$

thus, $x_{-5} > x_{-3}$.
From equation (19) and (20), we obtain

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-1}x_{-5}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}} > \quad (25)$$

$$\frac{1+x_{-1}x_{-3}x_{-5}x_{-7}x_{-9}}{x_{-3}x_{-5}x_{-7}x_{-9}x_{-11}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+5}\frac{1}{1+x_{2i-9}x_{2i-7}x_{2i-5}x_{2i-3}x_{2i-1}}$$

thus, $x_{-3} > x_{-1}$.
We, therefore, here $x_{-11} > x_{-9} > x_{-7} > x_{-5} > x_{-3} > x_{-1}$.We arrive at a conradiction.
Suppose that $a_2 = a_4 = a_6 = a_8 = a_{10} = a_{12} = 0$. From that equation (26) in (e) follows. It is noted that since proof of equation (21) is similar, it is omitted here.

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-2}x_{-4}x_{-6}x_{-8}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} > \tag{26}$$

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-2}x_{-4}x_{-6}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}$$

thus, $x_{-10} > x_{-8}$.
From that equation (27) in (e) follows. Proof of equation (22) is similar and will be omitted.

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-2}x_{-4}x_{-6}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+1}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} > \tag{27}$$

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-2}x_{-4}x_{-8}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}$$

thus, $x_{-8} > x_{-6}$.
From that equation (28) in (e) follows. Proof of equation (23) is similar and will be omitted.

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-2}x_{-4}x_{-8}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+2}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} > \tag{28}$$

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-2}x_{-6}x_{-8}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}$$

thus, $x_{-6} > x_{-4}$.
From that equation (29) in (e) follows. Proof of equation (24) is similar and will be omitted.

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-2}x_{-6}x_{-8}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+3}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} > \tag{29}$$

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-4}x_{-6}x_{-8}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}$$

thus, $x_{-4} > x_{-2}$.
From that equation (30) in (e) follows. Proof of equation (25) is similar and will be omitted.

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_0x_{-4}x_{-6}x_{-8}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+4}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}} > \quad (30)$$

$$\frac{1+x_0x_{-2}x_{-4}x_{-6}x_{-8}}{x_{-2}x_{-4}x_{-6}x_{-8}x_{-10}} = \sum_{j=0}^{\infty}\prod_{i=1}^{6j+5}\frac{1}{1+x_{2i-8}x_{2i-6}x_{2i-4}x_{2i-2}x_{2i}}$$

thus, $x_{-2} > x_0$.
We obtain here $x_{-10} > x_{-8} > x_{-6} > x_{-4} > x_{-2} > x_0$.We arrive at a conradiction which completes the proof of the theorem.

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