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Solving nonlinear fifth-order boundary value problems by differential transformation method

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Summary. In this study, we implement differential transformation method (DTM) for solving nonlinear fifth-order boundary value problems arising in viscoelastic flows. A numerical example is presented to illustrate the efficiency and reliability of this method.

Key words: Fifth-order boundary value problems, differential transformation method, numerical solution

1. Introduction

In this work, we consider a fifth-order boundary value problem of the form

$$y^{(5)}(x) = f(x,y), 0 < x < b, \tag{1.1}$$

subject to the boundary conditions

$$y(0) = A_0, y'(0) = A_1, y''(0) = A_2, y(b) = B_0, y'(b) = B_1. \tag{1.2}$$

where is $f(x,y)$ is a given continuous, nonlinear function of $y = y(x)$ and $A_i(i = 0,1,2)$, and $B_i(i = 0,1)$ are real finite constants [1].
Such problems arise in the mathematical modeling of viscoelastic flows [2,3]. Theorems which list the conditions for the existence and uniqueness of solutions of this types of problems are thoroughly discussed; see Agarwal [4].
In [2,3], two numerical algorithms,namely,spectral Galerkin methods and spectral collocation methods were applied to address the numerical issues related

to such problems. Moreover, Khan [5] and Wazwaz [6] investigated these problems by using the finite-difference methods and decomposition method, respectively.Recently, Caglar et al.[1], Siddiqi and Akram[7] investigated the fifth order boundary value problems using spline functions.
In the present paper, differential transformation method has been used to solve fifth-order boundary value problems which are assumed to have a unique solution in the interval of integration [4]. The concept of differential transform was first introduced by Zhou [8], in a study about electric circuit analysis. It is a semi-numerical-analytic-technique that formulizes Taylor series in a totally different manner. With this technique, the given differential equation and related boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution.

2. **Differential transformation method**

The differential transformation of the kth derivative of a function $f(x)$ is as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}, \tag{2.1}$$

and the inverse differential transformation is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{2.2}$$

The following theorems that can be deduced from Eqs. (2.1) and (2.2) are given below:
In actual applications, function $f(x)$ is expressed by a finite series and Eq. (2.2) can be written as

$$f(x) = \sum_{k=0}^{n} F(k)(x - x_0)^k. \tag{2.3}$$

Eq. (2.3) implies that $\sum_{k=n+1}^{\infty} F(k)(x - x_0)^k$ is negligibly small. In fact, n is decided by the convergence of natural frequency in this study.

Theorem 1 If $f(x) = g(x) \pm h(x)$,then $F(k) = G(k) \pm H(k)$.

Theorem 2 If $f(x) = ag(x)$,then $F(k) = aG(k)$,where a is a constant.

Theorem 3 If $f(x) = \frac{d^m g(x)}{dx^m}$,then $F(k) = \frac{(m+k)!}{k!} G(m + k)$.

Theorem 4 If $f(x) = g(x)h(x)$,then $F(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$.

Theorem 5 If $f(x) = x^n$,then $F(k) = \delta(k-n)$ where, $\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n. \end{cases}$

3. Numerical results

Example 3.1. Consider the following boundary value problem

$$y^{(5)}(x) = e^{-x}y^2(x), 0 < x < 1, \tag{3.1}$$

subject to the boundary conditions

$$\begin{array}{l} y(0) = y'(0) = y''(0) = 1, \\ y(1) = y'(1) = e. \end{array} \tag{3.2}$$

Theoretical solution for this problem is

$$y(x) = e^x. \tag{3.3}$$

Knowing that the differential transformation of e^{-x} is $(-1)^k/k!$ and using Theorems 3 and 4, Eq. (3.1) is transformed as follows:

$$\begin{array}{l} Y(k+5) = \frac{1}{(k+1)(k+2)(k+3)(k+4)(k+5)} \times \\ \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} \frac{(-1)^{k-k_2}}{(k-k_2)!} Y(k_1)Y(k_2-k_1). \end{array} \tag{3.4}$$

By using Eqs.(2.1) and (3.2), the following transformed boundary conditions at $x_0 = 0$ can be obtained:

$$Y(0) = 1, Y(1) = 1, Y(2) = \frac{1}{2}, \sum_{k=0}^{n} Y(k) = e \text{ and } \sum_{k=1}^{n} kY(k) = e, \tag{3.5}$$

where, n is a sufficiently large integer. By using the inverse transformation rule in Eq.(2.2), for $n = 7$, we get

$$y(x) = 1 + x + \frac{x^2}{2} + ax^3 + bx^4 + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + O(x^8), \tag{3.6}$$

where, according to Eq.(2.1),

$$a = \frac{y'''(0)}{3!} = Y(3) \text{ and } b = \frac{y^{(4)}(0)}{4!} = Y(4).$$

We can obtain the following system of equations from the fourth and fifth equalities in Eq. (3.5) :

$$a + b = e - \frac{1265}{504}, \quad 3a + 4b = e - \frac{1477}{720}. \tag{3.7}$$

This in turn gives

$$a = 0.1665518346 \text{ and } b = 0.0418093590. \tag{3.8}$$

Accordingly, we get the following series solution

$$\begin{aligned} y(x) = {} & 1 + x + 0.5x^2 + 0.1665518346x^3 + 0.0418093590x^4 \\ & + 0.0083333333x^5 + 0.0013888889x^6 + 0.0001984127x^7 \\ & + O(x^8). \end{aligned} \tag{3.9}$$

By continuing the same procedure for $n = 13$, we get the following series solution:

$$\begin{aligned} y(x) = {} & 1 + x + 0.5x^2 + 0.1666666665x^3 + 0.0416666668x^4 \\ & + 0.0083333333x^5 + 0.0013888889x^6 + 0.0001984127x^7 \\ & + 0.0000248016x^8 + 0.0000027557x^9 + 0.0000002756x^{10} \\ & + 0.0000000251x^{11} + 0.0000000021x^{12} + 0.000000000x^{13} \\ & + O(x^{14}). \end{aligned} \tag{3.10}$$

Numerical results for $n = 7$ and $n = 13$ with the comparison to the exact solution $y(x) = e^x$ are given in Table 1.

Table 1: Numerical results compared to the exact solution for Example 3.1.

x	DTM($n = 7$)	DTM($n = 13$)	Exact
0.0	1.0	1.0	1.0
0.1	1.1051708175	1.1051709181	1.1051709181
0.2	1.2214020677	1.2214027582	1.2214027582
0.3	1.2214020677	1.3498588076	1.3498588076
0.4	1.4918209843	1.4918246976	1.4918246976
0.5	1.6487157324	1.6487212707	1.6487212707
0.6	1.8221120435	1.8221188004	1.8221188004
0.7	2.0137460312	2.0137527075	2.0137527075
0.8	2.2255360184	2.2255409285	2.2255409285
0.9	2.4596011705	2.4596031112	2.4596031112
1.0	2.7182818285	2.7182818285	2.7182818285

As one can see from Table 1, the results obtained with the differential transformation method for $n = 13$ are ten digits accurate. Also, as the number of terms involved increase, one can observe that the series solution obtained by the differential transformation method converges to the series expansion of the exact solution $y(x) = e^x$.

4. Conclusion

Differential transformation method (DTM) was applied to the nonlinear fifth-order boundary value problems. The study showed that this method is simple and easy to use and produces reliable results.

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