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Control of a Double-Inverted Pendulum For Nonlinear System

Mustafa Demirci

Gaziosmanpasa University Tokat Occupational School Tasliciftlik Campus, 60100 Tokat, Turkey
e-mail: mdemirci34@yahoo.com

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Summary. The aim of the paper is to compare two different techniques to design a stabilizing control the double inverted pendulum in an upright position. These techniques are the Pole Placement and Linear Quadratic Regulator. Techniques have been used to design controls for the nonlinear system. For our double pendulum system, it would be desirable to keep the values of the state variables small as a large displacement from the origin may cause the nonlinear system to be unstable we would also interested in the control that gives the largest region of stability for the system, as we then could allow a larger set of perturbations from upright position.

Key words: Nonlinear , Stability, Pole Placement,Linear Quadratic Regulator and Double-Inverted Pendulum

1. Introduction

To design a stabilizing controller for a single inverted pendulum is a typical problem in control system design based on the state space approach [1] [2] and is known as a good analogy to the design of a controller for launching a rocket. A stabilize pendulum is useful to show the power of a control mechanism to laymen of the state space theory.

This paper follows up work carried out in a recent publication [10] by the same author, who dealt with the design of feedback controllers for linear system with applications to control of a double-inverted pendulum. Previously, the problem of stabilizing the double inverted pendulum for the linear system had been solved using two different techniques that were the Pole Placement and Linear Quadratic Regulator to design for the system.

The current paper focuses on the nonlinear system, to compare same techniques to design a stabilizing controller for the double inverted pendulum in an upright position. The pendulum system we will investigate is based on the system described in [3] with the assumptions that the double inverted pendulum moves on a horizontal rail.

2.Mathematical Model for Double Inverted Pendulum

The pendulum system we will take the double inverted pendulum as described in [3] with the assumption. The double inverted pendulum moves on a horizontal rail is schematically shown in Figure 1. The system is composed of:
A double inverted pendulum consisting of two aluminum rods, which are connected by a hinge, and the lower rod connected to a cart by a second hinge. The motion of the rods is smooth and is restricted to the vertical plane containing the double pendulum.
·A cart moves on a horizontal rain.
·The cart-driving system consisting of a motor, a pulley and belt transmission system using a timing belt and power amplifier.

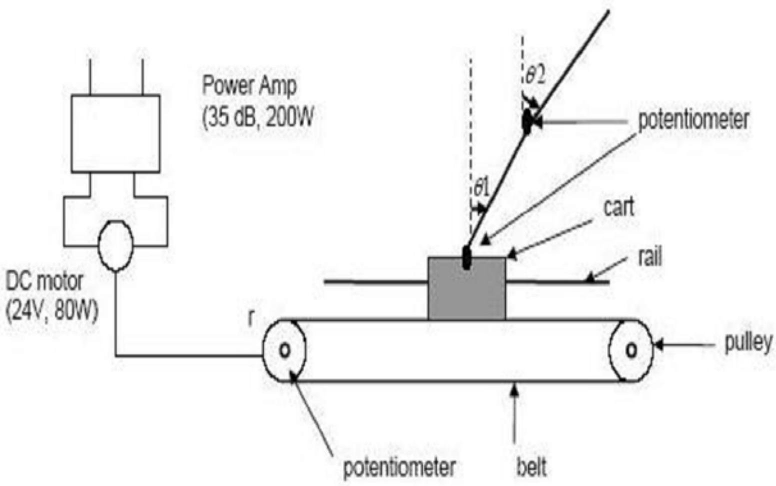

Figure.1. Diagram of the Double Inverted Pendulum System
Some assumptions were given by *Furute et al* (1978) in the mathematical model of the system.
·Each pendulum is rigid body
·The length of the belt does not change during the experiment.
·The driving force to the cart is directly applied to the cart without delay and is proportional to the input to the amplifier.
·The friction force against the motion of the cart, and frictional force generated at the connecting hinges is proportional to the difference the angular velocities of the upper and lower pendulums.
The above assumptions were given by *Furute et al.* A mathematical model can be derived on the Lagrange equation according to the above assumptions. But

the kinetic energy, potential energy and dissipation energy will be given here for our case of the motion of the double inverted pendulum on a horizontal rail, assuming that subscripts 1 and 2 denote the lower and upper pendulums respectively.

(1) For the Lower Pendulum:

Kinetic Energy

$$K_1 = \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} m_1 \left\{ \left[\frac{d}{dt} (l_1 \sin\theta_1) \right]^2 + \left[\frac{d}{dt} (l_1 \cos\theta_1) \right]^2 \right\}$$

Potential Energy

$$P_1 = m_1 g l_1 \cos\theta_1$$

Dissipation Energy

$$D_1 = \frac{1}{2} c_1 \dot{\theta}_1^2$$

(2) For the upper pendulum:

Kinetic Energy

$$K_2 = \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \left\{ \left[\frac{d}{dt} (L \sin\theta_1 + l_2 \sin\theta_2) \right]^2 + \left[\frac{d}{dt} (L \cos\theta_1 + l_2 \cos\theta_2) \right]^2 \right\}$$

Potential Energy

$$P_2 = m_2 g (L \cos\theta_1 + l_2 \cos\theta_2)$$

Dissipation Energy

$$D_2 = \frac{1}{2} c_1 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 \right)$$

(3) For the cart:

Kinetic Energy

$$K_3 = \frac{1}{2} M \dot{r}^2$$

Potential Energy

$$P_3 = 0$$

Dissipation Energy

$$D_3 = \frac{1}{2} F \dot{r}^2$$

where r is the distance of the cart from the reference position, $\theta_1 \theta_2$ are the angles of the lower pendulum and the upper pendulum from the vertical line,

m_i denote the mass, l_i the distance between the center of the hinge and its center of gravity, J_i is moment of inertia and c_i is the friction coefficient for rotation between the pendulum and hinge. L is the length (between the hinges) of the lower pendulum. M, F are the mass and friction coefficients of the cart and g is the acceleration of gravity.

$$K = \sum_{i=1}^{3} K_i, P = \sum_{i=1}^{3} P_i, D = \sum_{i=1}^{3} D_i$$

Then the following relations exist (summing the kinetic, potential and dissipation energies)

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{r}}\right) - \frac{\partial K}{\partial r} + \frac{\partial P}{\partial r} + \frac{\partial D}{\partial r} &= ae \\ \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_i}\right) - \frac{\partial K}{\partial \theta_i} + \frac{\partial P}{\partial \theta_i} + \frac{\partial D}{\partial \dot{\theta}_i} &= 0 \quad i = 1, 2 \end{aligned} \tag{1}$$

where a is the gain of overall cart driving system and e is the input to voltage to the amplifier satisfying $e_0 \geqslant |e|$.

$$\bar{x} = [r, \theta_1, \theta_2]^T, u\ e_0 = e \text{ and (1) is writtenas}$$

$$K_1 \bar{x}'' = K_2 \bar{x}' + k_3 + k_4 u$$

where

$$K_1 = \begin{vmatrix} m_1 + m_2 + M & (m_1 l_1 + m_2 L)\cos\theta_1 & m_2 l_2 \cos\theta_2 \\ (m_1 l_1 + m_2 L)\cos\theta_1 & J_1 + m_1 l_1^2 + m_2 L^2 & m_2 l_2 L \cos(\theta_1 - \theta_2) \\ m_2 l_2 \cos\theta_2 & m_2 l_2 L \cos(\theta_1 - \theta_2) & J_2 + m_2 l_2^2 \end{vmatrix}$$

$$K_2 = \begin{vmatrix} -F & (m_1 l_1 + m_2 L)\dot{\theta}_1 \sin\theta_1 & m_2 l_2 \dot{\theta}_2 \sin\theta_2 \\ 0 & -c_1 - c_2 & -m_2 l_2 L \dot{\theta}_2 \sin(\theta_1 - \theta_2) + c_2 \\ 0 & -m_2 l_2 L \dot{\theta}_1 \sin(\theta_1 - \theta_2) + c_2 & c_2 \end{vmatrix}$$

$$k_3 = \begin{vmatrix} 0 \\ (m_1 l_1 + m_2 L)\, g \sin\theta_1 \\ m_2 l_2 g \sin\theta_2 \end{vmatrix} \qquad k_4 = \begin{vmatrix} e_0 a \\ 0 \\ 0 \end{vmatrix}$$

Now we define

$$x = \begin{vmatrix} \bar{x} \\ \bar{x}' \end{vmatrix}$$

Then

$$x' = \begin{vmatrix} \bar{x}' \\ K_1^{-1}\left(K_2\bar{x}' + k_3 + k_4 u\right) \end{vmatrix}$$

gives the mathematical model for the double inverted pendulum system, where $\dot{x} = x'$.
To analyze non-linear systems is much difficult than linear systems. One of the principal techniques for the analysis of nonlinear systems is to approximate or bound them by appropriate linear systems, and then use linear theory. The nonlinear case is different in two essential respects. First, since equilibrium points are solutions, in this case, to nonlinear equations, finding such solutions is somewhat more of an accomplishment than in the linear case. Thus, a description of equilibrium points often constituter significant information. Second, and perhaps more fundamentally, the equilibrium point distribution points in virtually any special pattern in state space.

3.Stability

We consider the nonlinear system $\dot{x} = f(x)$ where f is continuously differentiable function.

Definition 1: $\dot{x} = f(x)$ has an equilibrium at $\bar{x}$, so that $f(\bar{x}) = 0$.

Definition 2: The equilibrium is stable if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x(0) - \bar{x}| < \delta$ implies $|x(t) - \bar{x}| < \varepsilon$ for all $t > 0$.

Definition 3: $\bar{x}$ is an asymptotically stable equilibrium point if it is stable and convergent. *i.e* : If there exists $\Delta > 0$ such that $|x(0) - \bar{x}| < \Delta$ implies that $x(t) \to \bar{x}$ as $t \to \infty$.

Definition 4: $\bar{x}$ is an unstable equilibrium point if it is not stable

Definition 5: Let us consider a nonlinear system such as $\dot{x} = f(x)$ and expand it into the form $\dot{x} = Ax(t) + g(x)$ where $g(x)$ contains all the higher powers of x to the extent that

$$\lim_{\|x\| \to 0} = \frac{\|g(x)\|}{\|x\|} = 0$$

$g(x)$ goes to zero faster than x does. Liapunov has shown that such a system will be stable if all the eigenvalues of A are strictly inside the left hand part.
Now, let $\dot{x} = f(x)$, $x(0) = x_0$ and $\bar{x}$ equilibrium point then we can write

$$\dot{x} = f(\bar{x}) + A(x - \bar{x}) + g(x - \bar{x})$$

where $A \in \Re^{nxn}$ and represents higher powers of $(x - \bar{x})$ If *Definition 5* is used, we have

$$\lim_{\|x-\bar{x}\|\to 0} = \frac{\|g\,(x-\bar{x})\|}{\|x-\bar{x}\|} = 0$$

Hence $g(x-\bar{x}) \to 0$ faster than $(x-\bar{x}) \to 0$ and so the higher powers in $(x-\bar{x})$ can be neglected,

$$f(x) - f(\bar{x}) = A(x-\bar{x})$$

$$\dot{x} - \dot{\bar{x}} = A(x-\bar{x}).$$

If we recall $x' = x - \bar{x}$, we get

$$\dot{x}' = Ax', x'(0) = x_0 - \bar{x}$$

the linearisation of the nonlinear system.

According to Liapunov's first method, which says that the linearized system is asymptotically stable at the origin, $\bar{x}$ is asymptotically stable for the nonlinear system.

We now consider the control system $\dot{x} = f(x, u)$, we have the conditions that is $f(\bar{x}, u) = 0$ and the equilibrium point of the system is $(\bar{x}, 0)$ .If we expand as before, we have

$$\dot{x} = f(\bar{x}, 0) + A(x-\bar{x}) + Bu + g(x-\bar{x}, u)$$

we can ignore the higher powers as before and the linearization of the nonlinear controlled system is

$$\dot{x}' = Ax' + Bu, x'(0) = x_0 - \bar{x} \tag{2}$$

If we return to Liapunov's first method, we have to get a system as $\dot{x}' = Sx'$ where S is system matrix. If u is taken as Fx' and substituting into (2), we get

$$\dot{x}' = Ax' + BFx' \tag{3}$$

$$\dot{x}' = (A+BF)x', x'(0) = x_0 - \bar{x},$$

where (A+BF) is system matrix and we can say that the linearization of the controlled system (3) approximates the nonlinear system at the point $\bar{x}$.

The nonlinear system can be thought of a perturbation of the linear system. If we can take a linear system of the form given by (3) and there is a feedback control, which brings the system back to the upright position, the same control will also work in the nonlinear case for small perturbations around the equilibrium position.

$$A = \begin{bmatrix} 0 & I_3 \\ L_1^{-1}k_3' & L_1^{-1}L_2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ L_1^{-1}k_4 \end{bmatrix}$$

$$k_3' = \begin{vmatrix} 0 & 0 & 0 \\ 0 & (m_1 l_1 + m_2 L)\, g & 0 \\ 0 & 0 & m_2 l_2 g \end{vmatrix}, k_4 = \begin{vmatrix} e_0 \\ 0 \\ 0 \end{vmatrix},$$

$$L_1|_{\theta_1=\theta_2=0} = \begin{bmatrix} m_1 + m_2 + M & m_1 l_1 + m_2 L & m_2 l_2 \\ m_1 l_1 + m_2 L & J_1 + m_1 l_1^2 + m_2 L^2 & m_2 l_2 L \\ m_2 l_2 & m_2 l_2 & J_2 + m_2 l_2^2 \end{bmatrix}$$

$$L_2|_{\theta_1=\theta_2=\dot{\theta}_1=\dot{\theta}_2=0} = \begin{bmatrix} -F & 0 & 0 \\ 0 & -c_1 - c_2 & c_2 \\ 0 & c_2 & -c_2 \end{bmatrix}$$

For the analysis and synthesis of a control system the following equivalent system is used

$$\dot{x} = Ax + Bu$$

$$y = Cx = [I_3 0]\, x$$

where I_3 is the 3x3 identity matrix, the vector x is the state, the 3-vector y is the output and the scalar u is the input of the system.

$$[A : B] = \begin{vmatrix} 0 & I_3 \\ A_{21} & A_{22} \end{vmatrix} \begin{matrix} 0 \\ B_2 \end{matrix} \Bigg|$$

where $A_{21} = L_1^{-1} k_3, A_{22} = L_1^{-1} L_2, B_2 = L_1^{-1} k_4$.
Thus a mathematical model of the double inverted pendulum has been derived. Since parameters have been given by *Furuta et al*, we will take those numerical values in order to simulate the double pendulum system. Parameters that are given in Table 1:

Parameter	**Values**
m_1	0.103 kg
m_2	0.070 kg
M	0.574 kg
l_1	0.225 m
l_2	0.177m
L	0.379 m
j_1	2.386×10^{-3} kg m^2
j_2	1.527×10^{-3} kg m^2
c_1	1.920×10^{-3} kg m^2 s^{-1}
c_2	8.930×10^{-4} kg m^2 s^{-1}
F	2.81 kg s^{-1}
a	46.7 N V^{-1}
e_0	0.3 V
g	9.8 m s^{-2}

Table 1: Parameters for the double inverted pendulum system

By substituting all parameters in the linearised model of the system we have

$$A_{21} = \begin{vmatrix} 0 & -2.3318 & 0.0672 \\ 0 & 48.3079 & -13.2645 \\ 0 & -53.2131 & 49.1599 \end{vmatrix}, \quad A_{22} = \begin{vmatrix} -4.6810 & 0.0140 & -0.0048 \\ 13.4514 & -0.3765 & 0.1861 \\ -1.5558 & 0.6688 & -0.4591 \end{vmatrix}$$

$$B_2 = \begin{vmatrix} 23.0889 \\ -67.0654 \\ 7.7568 \end{vmatrix}$$

Since the eigenvalues of the matrix A are

$$[0, 8.2299, 4.5047, -9.3104, -5.6993, -3.1914]$$

We have two positive eigenvalues and one zero eigenvalue. Therefore, the system is unstable.
The system is contrable, because of the rank of the matrix is equal 6 to for the double pendulum system. Hence, we have that the double inverted pendulum system is controllable and observable. Therefore we are able to stabilize the system and design a dynamic observer for the system.
Full details of Feedback Control, Dynamical Observers, the Pole Placement and the Linear Quadratic Regulator are given in [10].

4.Nonlinear System

Euler's method will be used to solve the nonlinear system for approximating the differential equation $\dot{x} = f(t, x)$ then an approximation solution to this equation is found as follows:

$$x(t_{n+1}) = x(t_n) + \Delta f(x(t_n)) \qquad n = 1, 2, 3, ...$$

where $x(t_0) = x_0$ and $\Delta = t_{n+1} - t_n$ is the step size.

5.Nonlinear

This solves for the nonlinear system with feedback $u = -Fx$, and intial condition is x_0:

$$\dot{x} = x' \qquad \bar{x}' = \left[\dot{r}, \dot{\theta}_1, \dot{\theta}_2\right]^T,$$

$$x' = \begin{vmatrix} \bar{x}' \\ K_1^{-1}\left(K_2\bar{x}' + k_3 + k_4 u\right) \end{vmatrix}$$

6.Nonlinear-observer

$x(t)$ is the state and $\hat{x}$ is the observer we will solve two equations:

$$x' = \begin{vmatrix} \bar{x}' \\ K_1^{-1}\left(K_2\bar{x}' + k_3 + k_4 u\right) \end{vmatrix}$$
$$\dot{\hat{x}} = TCx + (A - TC - BF)\hat{x}$$

where $u = F\hat{x}$

7.Nonlinear-observer error

We will solve two equations to get the error of the observer varies with time:

$$x' = \begin{vmatrix} \bar{x}' \\ K_1^{-1}\left(K_2\bar{x}' + k_3 + k_4 u\right) \end{vmatrix}$$

where $u = f(x + e)$

$$\dot{e} = (A - TC)e.$$

This equation expresses the error dynamics $\dot{e}$ of the estimate in terms of the system matrix A, the measurement matrix C and the observer matrix T. We see that if T is chosen so that $(A - TC)$ is a stable system matrix then e tends to zero. Indeed, the error tends to zero at a rate determined by the dominant eigenvalue of $(A - TC)$. The eigenvalues of this matrix are controlled by our choice of the matrix T.

8.Results for the Pole Placement Technique

The best control found by this technique was p6. The graph in Figure 2 shows the behavior of the nonlinear system with the control p6. Figure 3 shows the graph of G4, the control G4 had the largest radius of stability.
Figure 4 and 5 show the resulting behavior for the nonlinear systems with the observer.

9.The Linear Quadratic Regulator Method

The first control designed using this method used the matrix
$Q = diag\{0.001, 10, 10, 0.001, 0.001\}$ the weight the state variables.
Experience gained in using the pole placement method showed that if r were not controlled heavily, the system would then take a long time to damp. Investigation then continued to see how large the penalty on $r, \dot{r}, \theta_1, \theta_2$ should be. It was noticed that this method seems to minimize the oscillatory behavior of the system.
As has been shown in the graphs for the pole placement technique [10], the behavior of the linearsied system is extremely close to that of the nonlinear system for the small initial condition that was chosen. Therefore only the graphs that concern the nonlinear system are shown, in Figure 6,7,8 and 9.

10.Conclusion

Euler's method has been used to solve the nonlinear system. The main disadvantage with the Euler algorithm is the fixed integration step size Δ. Size of the step affects the degree of accuracy and also the stability properties of the calculated solution.
A smaller step size requires more calculation to be performed over the same time interval and so increases the time taken to computer solutions. Obviously, a step size must be chosen which minimizes the time for computation while providing an accurate behavior of the actual system. Typical step sizes used in the simulations were of the order of 10^{-2} or 10^{-3} seconds. Any smaller step size would result in a much longer run-time for the algorithms. Unfortunately, this means that the results may not be extremely accurate.
To design controls by the pole placement technique generate very small regions of stability for the nonlinear system and observer. Therefore the pole placement technique gave the least satisfactory results. The nonlinear model shows that the system is complicated and likely to be sensitive to small disturbance.

Controls for the double pendulum system-pole placement method

Control	Value	Stability Radius
• p1	[0.0001,-1.3929,0.8852,-0.1953,-0.0181,0.0251]	R1
f1	{-0.5+i,-0.5-i,-0.4+i,-0.4-i,-0.3+i,-0.3-i}	
• p2	[0.1422,-4.5554,5.2696,-0.0088,-0.1899,0.7251]	R2
f2	{-5+i,-5-i,-4+i,-4-i,-3+i,-3-i}	
• p3	[2.2427,-4.7837,9.4729,1.0313,0.2288,1.3570]	R3
f3	{-5+5i,-5-5i,-4+5i,-4-5i,-3+5i,-3-5i}	
• p4	[0.0000,-1.3535,0.8443,-0.1954,-0.0183,0.0237]	R4
f4	{-0.5+0.1i,-0.5-0.1i,-0.4+0.1i,-0.4-0.1i,-0.3+0.1i,-0.3-0.1i}	
• p5	[0.0032,-2.1437,1.7077,-0.1895,-0.1358,0.1972	R5
f5	{-1-i,-1+i,-2-i,-2+i,-3-i,-3+i}	
• p6	[1.1905,-7.8575,13.1042,0.9791,0.1138,1.9062]	R6
f6	{-4,-5,-6+i,-6-i,-7+i,-7-i}	

R6>R3>R2>R5>R1>R4

Table 2

Controls for the error of the observer-pole placement technique

Control	Value	Stability Radius
G1	1.5867 8.3985 -1.4473 2.1574 6.1633 4.1390 0.0573 2.5566 0.3968 -2.6869 51.1980 -52.1888 -0.3758 -2.9170 2.3901 0.2415 -18.9757 49.1431	r1
F1	{-1-0.6i,-1+0.6i,-2-0.4i,-2+0.4i,-3-0.2i,-3+0.2i}	
G2	4.2227 14.1538 -1.6527 20.0491 -62.5215 16.8172 -0.1457 0.7891 0.6478 -1.6256 46.3790 -52.7445 -0.3374 1.2226 0.3670 1.5865 -17.9409 50.5513	r2
F2	{-0.5-0.1i,-0.5+0.1i,-0.4-0.2i,-0.4+0.2i,-0.3-0.4i,-0.3+0.4i}	
G3	4.6150 3.7378 -0.1152 0.3379 21.2784 -1.6929 -1.2422 5.1641 -0.3582 -2.0159 45.9827 -49.7024 -0.1075 -2.8634 0.5543 1.5103 -20.2994 48.3589	r3
F3	{-0.2+0.1i,-0.2-0.1i,-5.5+0.3i,-5.5-0.3i,-2.2+0.7i,-2.2-0.7i}	
G4	5.0916 -1.5310 -0.5559 0.1149 -5.1746 -4.7951 0.1070 12.3892 0.1399 -2.4498 84.0033 -48.5860 -0.0308 -1.3379 12.0525 -0.0437 -20.1805 83.0760	r4
F4	{-4,-5,-6+i,-6-i,-7+i,-7-i}	
G5	-4.4063 13.3786 -1.5350 120.2897 -64.7206 16.6305 -0.0939 0.2065 0.6608 -0.1569 147.4765 -53.2417 0.0685 0.0629 0.5332 -1.1838 -12.3437 149.1224	r5
F5	{-0.5+10i,-0.5-10i,-0.3+10i,-0.3-10i,-0.1+10i,-0.1-10i}	
G6	-0.6454 26.7872 -1.6316 98.2856 121.7463 37.5568 -1.3036 17.6332 1.0860 30.6486 165.5361 -37.2035 0.5210 -2.3322 3.5456 -8.0754 -30.2134 52.5211	r6
F6	{-7+10i,-7-10i,-4+7i,-4-7i,-2+2i,-2-2i}	

r4>r6>r1>r2>r5>r3

Table 3

Controls for the double pendulum system linear quadratic regulator

Control	Value	Stability Radius
LQ1	[0.0316,-10.4654,15.1340,0.0030,-0.2592,2.0681] q=diag([0.001 10 10 0.001 0.001 0.001])	R1
LQ2	[0.3162 -10.7288 16.6885 0.4043 -0.1060 2.3250] q=diag([0.1 10 10 0.001 0.001 0.001])	R2
LQ3	[2.2361 -11.3747 22.7515 2.0518 0.5219 3.3185] q=diag([5 10 10 0.001 0.001 0.001])	R3
LQ4	[3.8730 -11.7705 27.2379 3.2605 0.9837 4.0467] q=diag([15 10 10 0.001 0.001 0.001])	R4
LQ5	[2.2361 -57.1210 79.3740 2.9483 -1.1052 11.3784] q=diag([5 10 10 1 10 0.1])	R5
LQ6	[2.2361 -75.0618 108.4011 4.4203 -0.9857 15.5966] q=diag([5 10 10 10 15 0.1])	R6

R3>R4>R5>R6>R2>R1

Table 4

Controls for the error of the observer linear quadratic regulator Technique

Control	Value	Stability Radius
LG1	-1.2736 -5.1980 -2.5755 -6.3276 8.4676 -28.5744 3.3226 10.0206 1.4588 -4.2600 62.4868 -27.6704 20.9363 11.4851 25.2668 23.8577 13.1196 256.9435 q=diag([0.001 10 10 0.001 0.001 0.001])	r1
LG2	7.7443 11.2074 5.4066 14.5475 127.1696 -11.2840 -4.1848 26.5143 6.1009 2.9783 258.1160 -64.8454 -0.1958 -1.1460 0.4331 1.5443 -32.7749 51.2899 q=diag([0.1 10 10 0.001 0.001 0.001])	r2
LG3	8.0187 40.5126 4.1356 59.5381 423.1552 22.5482 -0.6545 26.4130 2.1596 14.6601 238.3041 -39.8838 0.2021 -3.5041 2.6371 -2.5556 -44.7297 50.6790 q=diag([5 10 10 0.001 0.001 0.001])	r3
LG4	38.1883 5.0758 82.9691 2.8758 36.6780 24.2329 -0.7975 5.7602 -3.4810 -1.1848 51.4037 -50.7382 89.0496 16.2586 247.1662 7.0247 94.9818 121.0655 q=diag([5 10 10 10 15 0.1])	r4

r3>r2>r4>r1

Table 5

In this part, some graphs will be given which are simulations of the pendulum system and Table 6 is given which is the key for the graphs.

———	$r(m)$	$\dot{r}(ms^{-1})$
--------	$\theta_1(rad)$	$\dot{\theta}_1(rads^{-1})$
-·-·-·-	$\theta_2(rad)$	$\dot{\theta}_2(rads^{-1})$

Table 6

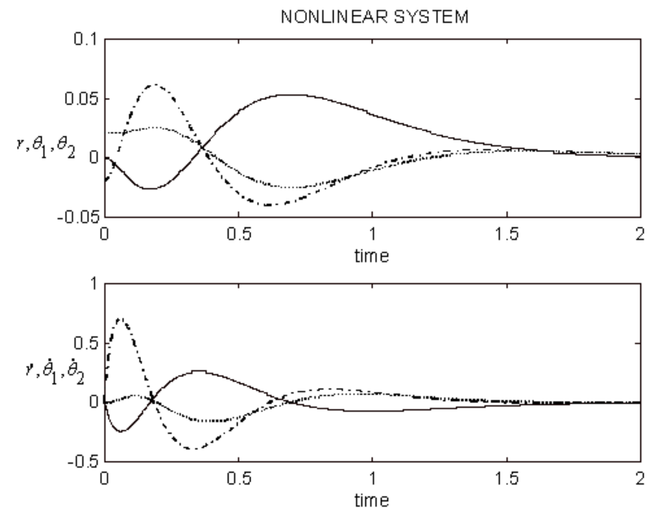

Figure 2: Results for p6 and f6 in Table 2

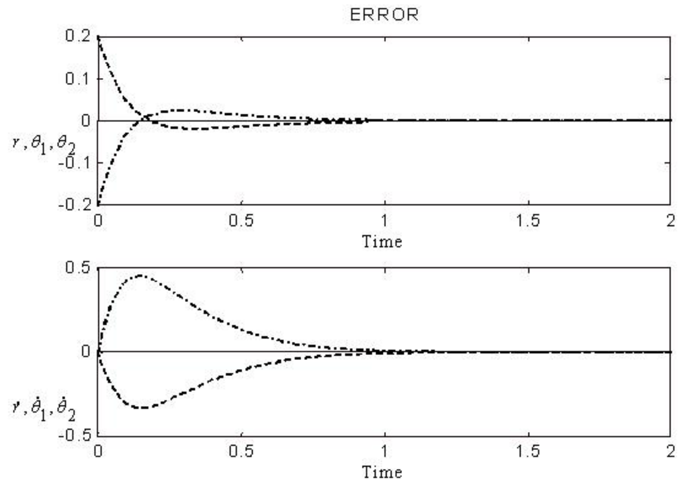

Figure 3: Results for G4 in Table 3

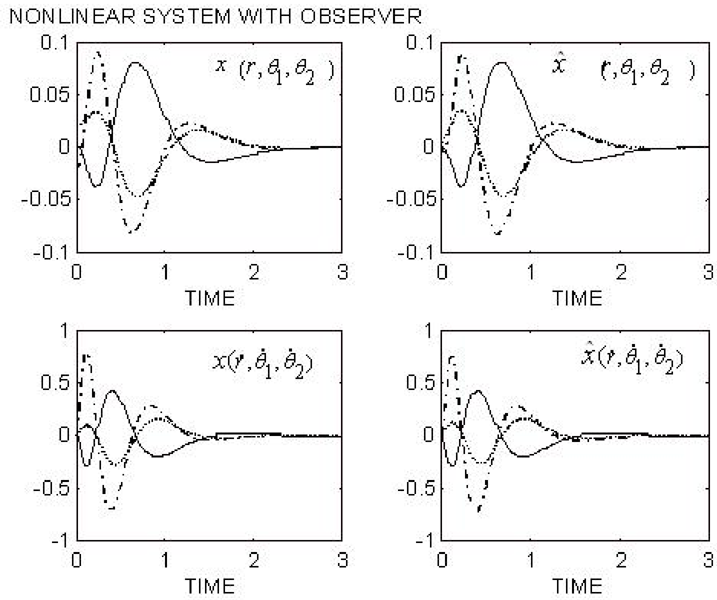

Figure 4: Results for p6 and G4 in Table 2 and 3

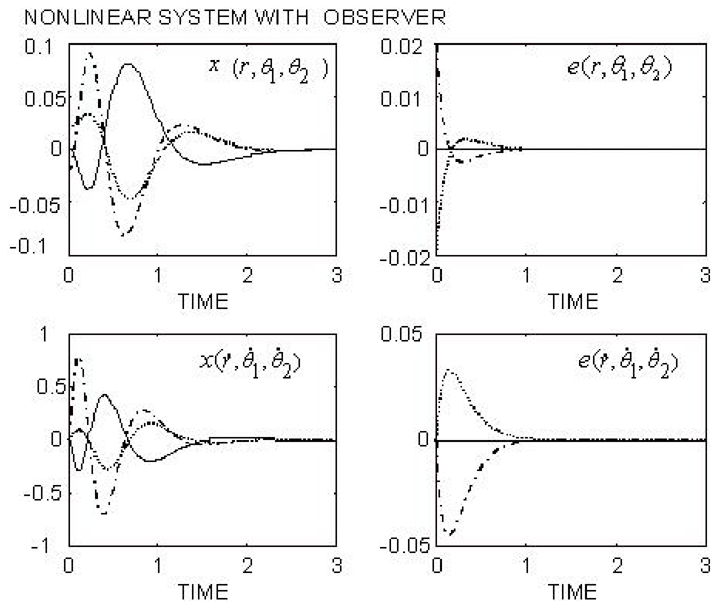

Figure 5: Results for p6 and G4 in Table 2 and 3

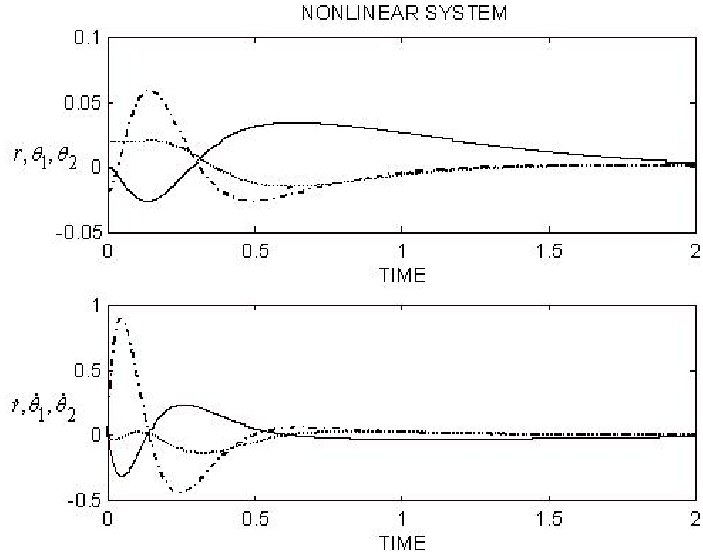

Figure 6: Results for LQ3 in Table 4

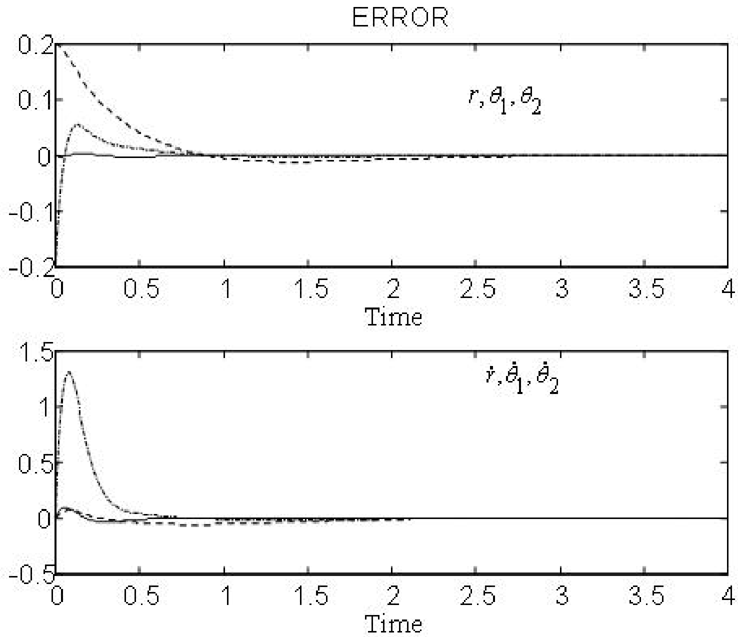

Figure 7: Results for LG3 in Table 5

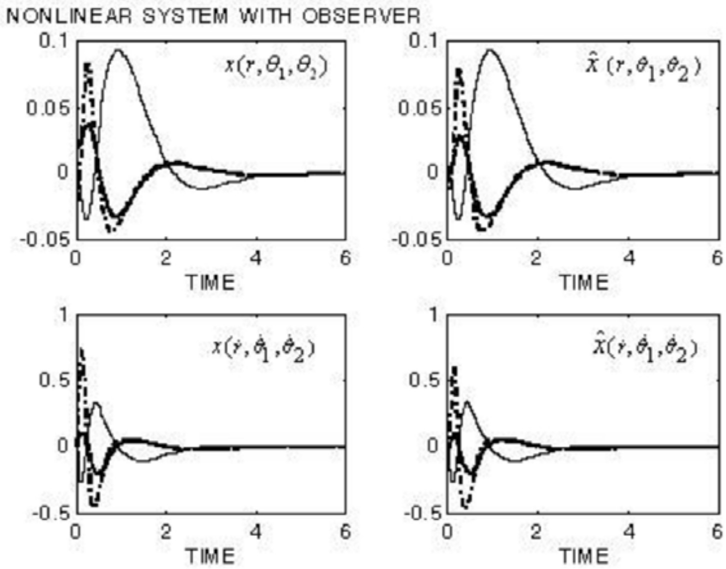

Figure 8: Results for LG3 and LQ3 in Table 4 and 5

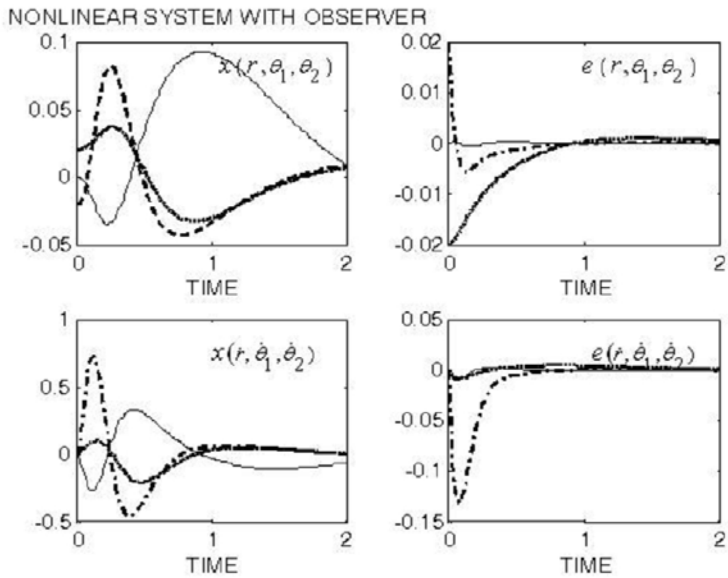

Figure9: Results for LG3 and LQ3 in Table 4 and 5

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