Selçuk J. Appl. Math.
Vol. 6. No.1. pp. 55-71, 2005

Selçuk Journal of Applied Mathematics

On the Solutions of the Difference Equation $x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{x_{n-1}}{B}\right\}$

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Received : April 26, 2005

Summary. In this paper we study the behaviour of the solutions of the following difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{x_{n-1}}{B}\right\}$$

where A, B and the initial conditions $x_{-1}$and x_0 are nonzero real numbers. In most of the cases we determine the behaviour of the solutions as a function of the parameters A, B and the initial conditions x_{-1} and x_0.

Key words: Difference equation, max operator, periodicity, behaviour.

1. Introduction

In this paper we study the behaviour of the solutions of the following difference equation

$$x_{n+1} = \max\left\{\frac{A}{x_n}, \frac{x_{n-1}}{B}\right\} \tag{1}$$

where A, B and the initial conditions $x_{-1}$and x_0 are nonzero real numbers.

Some closely related equations were investigated, in [1,2,3,4,5]. For example, the investigation of the difference equation

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, ..., \frac{A_k}{x_{n-k}}\right\},\ n = 0, 1, ... \tag{2}$$

where A_i, $i = 0, 1, ..., k$, are real numbers, such that at least one of the A_i and the initial conditions $x_0, x_{-1}, ..., x_{-k}$, are different from zero, was proposed in [3] and [4].

A special case of the max operator in (2) arises naturally in certain models in automatic control theory (see, [6,7]).

For some other recent studies concerning, the periodic nature of scalar nonlinear difference equations see, for example, [8,9,10].

2. Main Results

*2.1. Case I $B < 0 < A$.*In this section we consider the behaviour of the solutions of (1) in the case $B < 0 < A$. The following theorem completely desciribes the behaviour of the solutions of (1) in this case.

Theorem 1 *Consider (1),with $B < 0 < A$,*

a) If $0 < x_0, x_{-1}$, then

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, ...)$$

b) If $x_0, x_{-1} < 0$ and $\frac{x_0}{B} < \frac{AB}{x_{-1}}$, then

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, ...)$$

c) If $x_0, x_{-1} < 0$ and $\frac{AB}{x_{-1}} < \frac{x_0}{B}$, then

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B}, \frac{AB}{x_0}, ...)$$

d) If $x_{-1} < 0 < x_0$ and $x_1 = \frac{A}{x_0}$, then

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, x_0, ...)$$

e) If $x_{-1} < 0 < x_0$ and $x_1 = \frac{x_{-1}}{B}$, then

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, ...)$$

f) If $x_0 < 0 < x_{-1}$ and $x_1 = \frac{A}{x_0}$, $B < -1$, then

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B}, \frac{AB}{x_0}, ...)$$

g) If $x_0 < 0 < x_{-1}$ and $x_1 = \frac{A}{x_0}$, $-1 < B < 0$, then

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{A}{Bx_0}, Bx_0, \frac{A}{Bx_0}, Bx_0, ...)$$

h) If $x_0 < 0 < x_{-1}$ *and* $x_1 = \frac{x_{-1}}{B}$, $\frac{AB}{x_0} < \frac{x_{-1}}{B^2}$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{x_{-1}}{B^2}, \frac{AB^2}{x_{-1}}, \frac{x_{-1}}{B^2}, \frac{AB^2}{x_{-1}}, ...)$$

i) If $x_0 < 0 < x_{-1}$ *and* $x_1 = \frac{x_{-1}}{B}$, $\frac{Bx_{-1}}{2} < \frac{AB}{x_0}$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B}, \frac{AB}{x_0}, ...)$$

Proof. (a) Let $0 < x_0, x_{-1}$, then $0 < x_n$ for $-1 \leq n$ and $x_1 = \max\left\{\frac{A}{x_0}, \frac{x_{-1}}{B}\right\} = \dfrac{A}{x_0}$. Then, $x_1 = \dfrac{A}{x_0}$, $\frac{x_{-1}}{B} < 0 < \frac{A}{x_0}$ and $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0$. Hence by induction we get $x_{n+1} = \frac{A}{x_n}, 0 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, ...)$$

(b), (c) Let $x_0, x_{-1} < 0$, then $0 < x_n$ for $1 \leq n$ and $x_1 = \max\left\{\frac{A}{x_0}, \frac{x_{-1}}{B}\right\} = \frac{x_{-1}}{B}$. If $\frac{x_0}{B} < \frac{AB}{x_{-1}}$, then $x_1 = \frac{x_{-1}}{B}$ and $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{AB}{x_{-1}}$. It follows $x_3 = \max\left\{\frac{x_{-1}}{B}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B}$, $\frac{x_{-1}}{B^2} < 0 < \frac{x_{-1}}{B}$ and $x_4 = \max\left\{\frac{AB}{x_{-1}}, \frac{AB^2}{x_{-1}}\right\} = \frac{AB}{x_{-1}}$. By induction we obtain that $x_{2n} = \frac{AB}{x_{-1}}$ and $x_{2n-1} = \frac{x_{-1}}{B}$ for $1 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, ...)$$

If $\frac{AB}{x_{-1}} < \frac{x_0}{B}$, then $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{x_0}{B}$. It follows $x_3 = \max\left\{\frac{AB}{x_0}, \frac{x_{-1}}{B^2}\right\} = \frac{AB}{x_0}$ and $x_4 = \max\left\{\frac{x_0}{B}, \frac{x_0}{B^2}\right\} = \frac{x_0}{B}$. By induction we get $x_{2n} = \frac{x_0}{B}$ and $x_{2n+1} = \frac{AB}{x_0}$ for $1 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B}, \frac{AB}{x_0}, ...)$$

(d), (e) Let $x_{-1} < 0 < x_0$, then $0 < x_n$ for $0 \leq n$. If $x_1 = \frac{A}{x_0}$ or $\frac{x_{-1}}{B} < \frac{A}{x_0}$, then $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0$, $\frac{x_0}{B} < 0 < x_0$ and $x_3 = \max\left\{\frac{A}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{x_0}\max\left\{1, \frac{1}{B}\right\} = \frac{A}{x_0}$. Hence by induction it is easy to see that (1) implies the difference equation $x_{n+1} = \frac{A}{x_n}$ for $0 \leq n$, in this case, we write

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, x_0, ...)$$

If $x_1=\frac{x_{-1}}{B}$, then $x_2=\max\left\{\frac{A}{x_{-1}},\frac{x_0}{B}\right\}=\max\left\{\frac{AB}{x_{-1}},\frac{x_0}{B}\right\}=\frac{AB}{x_{-1}}$. It follows $x_3=\max\left\{\frac{x_{-1}}{B},\frac{x_{-1}}{B^2}\right\}=\frac{x_{-1}}{B}=x_1$ and $x_4=\max\left\{\frac{AB}{x_{-1}},\frac{A}{x_{-1}}\right\}=\frac{A}{x_{-1}}\min\{B,1\}=\frac{AB}{x_{-1}}$.Thus by induction we have $x_{2n}=\frac{AB}{x_{-1}}$ and $x_{2n-1}=\frac{x_{-1}}{B}$ for $1\le n$, that is,

$$(x_n)=(x_{-1},x_0,\frac{x_{-1}}{B},\frac{AB}{x_{-1}},\frac{x_{-1}}{B},\frac{AB}{x_{-1}},...)$$

(f), (g) Let $x_0<0<x_{-1}$ and $x_1=\frac{A}{x_0}<0$. Then, $x_2=\max\left\{x_0,\frac{x_0}{B}\right\}=\frac{x_0}{B}$, $0<\frac{x_0}{B}$ and $0<x_n$, $2\le n$. If $B<-1$ and $x_1=\frac{A}{x_0}$, then $x_2=\frac{x_0}{B}$ and $x_3=\max\left\{\frac{AB}{x_0},\frac{A}{Bx_0}\right\}=\frac{A}{x_0}\min\left\{B,\frac{1}{B}\right\}=\frac{AB}{x_0}$. By induction we get $x_{2n}=\frac{x_0}{B}$ and $x_{2n+1}=\frac{AB}{x_0}$ for $1\le n$, that is,

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},\frac{x_0}{B},\frac{AB}{x_0},\frac{x_0}{B},\frac{AB}{x_0},...)$$

If $B\in(-1,0)$, then $x_1=\frac{A}{x_0}$, $x_2=\frac{x_0}{B}$ and $x_3=\max\left\{\frac{AB}{x_0},\frac{A}{Bx_0}\right\}=\frac{A}{x_0}\min\left\{B,\frac{1}{B}\right\}=\frac{A}{Bx_0}$ and $x_4=\max\left\{Bx_0,\frac{x_0}{B^2}\right\}=x_0\min\left\{B,\frac{1}{B^2}\right\}=Bx_0$. By induction we get $x_{2n}=Bx_0$ and $x_{2n-1}=\frac{A}{Bx_0}$ for $2\le n$, that is,

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},\frac{x_0}{B},\frac{A}{Bx_0},Bx_0,\frac{A}{Bx_0},Bx_0,...)$$

(h), (i) Let $x_0<0<x_{-1}$ and $x_1=\frac{x_{-1}}{B}$, Hence we get $0<x_n$ for $2\le n$ and $x_2=\max\left\{\frac{AB}{x_{-1}},\frac{x_0}{B}\right\}=\frac{x_0}{B}$, $0<\frac{x_0}{B}$.

If $\frac{AB}{x_0}<\frac{x_{-1}}{B^2}$ and $x_1=\frac{x_{-1}}{B}$.Then, $x_2=\frac{x_0}{B}$ and $x_3=\max\left\{\frac{AB}{x_0},\frac{x_{-1}}{B^2}\right\}=\frac{x_{-1}}{B^2}$. It follows $x_4=\max\left\{\frac{AB^2}{x_{-1}},\frac{x_0}{B^2}\right\}=\frac{AB^2}{x_{-1}}$, $0<\frac{AB^2}{x_{-1}}$. $x_5=\max\left\{\frac{x_{-1}}{B^2},\frac{x_{-1}}{B^3}\right\}=\frac{x_{-1}}{B^2}$, $\frac{x_{-1}}{B^3}<0<\frac{x_{-1}}{B^2}$. By induction we get $x_{2n}=\frac{AB^2}{x_{-1}}$ and $x_{2n-1}=\frac{x_{-1}}{B^2}$ for $2\le n$, that is,

$$(x_n)=(x_{-1},x_0,\frac{x_{-1}}{B},\frac{x_0}{B},\frac{x_{-1}}{B^2},\frac{AB^2}{x_{-1}},\frac{x_{-1}}{B^2},\frac{AB^2}{x_{-1}},...)$$

If $\frac{x_{-1}}{B^2}<\frac{AB}{x_0}$ and $x_1=\frac{x_{-1}}{B}$.Then, $x_2=\frac{x_0}{B}$, $x_3=\max\left\{\frac{AB}{x_0},\frac{x_{-1}}{B^2}\right\}=\frac{AB}{x_0}$. It follows $x_4=\max\left\{\frac{x_0}{B},\frac{x_0}{B^2}\right\}=\frac{x_0}{B}=x_2$ and $x_5=\max\left\{\frac{AB}{x_0},\frac{A}{x_0}\right\}=\frac{A}{x_0}\min\{B,1\}=\frac{AB}{x_0}=x_3$. Hence by induction we obtain that $x_{2n}=\frac{x_0}{B}$ and $x_{2n-1}=\frac{AB}{x_0}$ for $2\le n$, that is

$$(x_n)=(x_{-1},x_0,\frac{x_{-1}}{B},\frac{x_0}{B},\frac{AB}{x_0},\frac{x_0}{B},\frac{AB}{x_0},...)$$

2.2. Case II $A<0<B$. In this section we consider the behaviour of the solutions of (1) in the case $A<0<B$. The following theorem completely describes the behaviour of the solutions of (1) in this case.

Theorem 2 *Consider (1),with $A<0<B$.*

a) If $0<x_0,x_{-1}$, then

$$(x_n)=(x_{-1},x_0,\frac{x_{-1}}{B},\frac{x_0}{B},\frac{x_{-1}}{B^2},\frac{x_0}{B^2},...,\frac{x_{-1}}{B^n},\frac{x_0}{B^n},...)$$

b) If $x_0,x_{-1}<0$ and $1<B$, then

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},\frac{x_0}{B},\frac{AB}{x_0},\frac{x_0}{B^2},\frac{AB^2}{x_0},...,\frac{AB^{n-1}}{x_0},\frac{x_0}{B^n},...)$$

c) If $x_0,x_{-1}<0$ and $0<B<1$, then

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},x_0,\frac{A}{Bx_0},Bx_0,...,\frac{A}{B^{n-1}x_0},B^{n-1}x_0,...)$$

d) If $x_{-1}<0<x_0$, $x_1=\frac{A}{x_0}$ and $1<B$, then

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},x_0,\frac{A}{Bx_0},Bx_0,...,\frac{A}{B^{n-1}x_0},B^{n-1}x_0,...)$$

e) If $x_{-1}<0<x_0$, $x_1=\frac{A}{x_0}$ and $0<B<1$, then

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},\frac{x_0}{B},\frac{AB}{x_0},\frac{x_0}{B^2},\frac{AB^2}{x_0},...,\frac{AB^{n-1}}{x_0},\frac{x_0}{B^n},...)$$

f) If $x_{-1}<0<x_0$, $x_1=\frac{x_{-1}}{B}$ and $1<B$, then

$$(x_n)=(x_{-1},x_0,\frac{x_{-1}}{B},\frac{AB}{x_{-1}},\frac{x_{-1}}{B^2},\frac{AB^2}{x_{-1}},...,\frac{x_{-1}}{B^n},\frac{AB^n}{x_{-1}},...)$$

g) If $x_{-1}<0<x_0$, $x_1=\frac{x_{-1}}{B}$, $\frac{AB}{x_{-1}}<\frac{x_0}{B}$ and $0<B<1$, then

$$(x_n)=(x_{-1},x_0,\frac{x_{-1}}{B},\frac{x_0}{B},\frac{AB}{x_0},\frac{x_0}{B^2},\frac{AB^2}{x_0},...,\frac{AB^{n-1}}{x_0},\frac{x_0}{B^n},...)$$

h) If $x_{-1}<0<x_0$, $x_1=\frac{x_{-1}}{B}$, $\frac{x_0}{B}<\frac{AB}{x_{-1}}$ and $0<B<1$, then

$$(x_n)=(x_{-1},x_0,\frac{x_{-1}}{B},\frac{AB}{x_{-1}},\frac{x_{-1}}{B},\frac{A}{x_{-1}},x_{-1},\frac{A}{Bx_{-1}},Bx_{-1},...,\frac{A}{B^{n-2}x_{-1}},B^{n-2}x_{-1},...)$$

i) If $x_0<0<x_{-1}$, $x_1=\frac{A}{x_0}$ and $1<B$, then

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},\frac{x_0}{B},\frac{AB}{x_0},\frac{x_0}{B^2},\frac{AB^2}{x_0},...,\frac{AB^{n-1}}{x_0},\frac{x_0}{B^n},...)$$

j) If $x_0 < 0 < x_{-1}$*,* $x_1 = \frac{A}{x_0}$ *and* $0 < B < 1$*, then*

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{Bx_0}, Bx_0, ..., \frac{A}{B^{n-1}x_0}, B^{n-1}x_0, ...)$$

k) If $x_0 < 0 < x_{-1}$*,* $x_1 = \frac{x_{-1}}{B}$, $\frac{x_0}{B} < \frac{AB}{x_{-1}}$ *and* $1 < B$*, then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B}, \frac{A}{x_{-1}}, x_{-1}, \frac{A}{Bx_{-1}}, Bx_{-1}, ..., \frac{A}{B^{n-2}x_{-1}}, B^{n-2}x_{-1}, ...)$$

l) If $x_0 < 0 < x_{-1}$*,* $x_1 = \frac{x_{-1}}{B}$, $\frac{AB}{x_{-1}} < \frac{x_0}{B}$ *and* $1 < B$*, then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B^2}, \frac{AB^2}{x_0}, ..., \frac{x_0}{B^n}, \frac{AB^n}{x_0}, ...)$$

m) If $x_0 < 0 < x_{-1}$*,* $x_1 = \frac{x_{-1}}{B}$ *and* $0 < B < 1$*, then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B^2}, \frac{AB^2}{x_{-1}}, ..., \frac{x_{-1}}{B^n}, \frac{AB^n}{x_{-1}}, ...)$$

Proof. (a) Let $0 < x_0, x_{-1}$, then $0 < x_n$ for $-1 \le n$ and $x_1 = \max\left\{\frac{A}{x_0}, \frac{x_{-1}}{B}\right\} = \frac{x_{-1}}{B}$, $\frac{A}{x_0} < 0 < \frac{x_{-1}}{B}$, then $x_1 = \frac{x_{-1}}{B}$ and it follows $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{x_0}{B}$, $x_3 = \max\left\{\frac{AB}{x_0}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B^2}$. Therefore $x_4 = \max\left\{\frac{AB^2}{x_{-1}}, \frac{x_0}{B^2}\right\} = \frac{x_0}{B^2}$. By induction we obtain that $x_{2n} = \frac{x_0}{B^n}$ and $x_{2n-1} = \frac{x_{-1}}{B^n}$ for $0 \le n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{x_{-1}}{B^2}, \frac{x_0}{B^2}, ..., \frac{x_{-1}}{B^n}, \frac{x_0}{B^n}, ...)$$

(b), (c) Let $x_{-1}, x_0 < 0$, then $x_1 = \max\left\{\frac{A}{x_0}, \frac{x_{-1}}{B}\right\} = \frac{A}{x_0}$, $\frac{x_{-1}}{B} < 0 < \frac{A}{x_0}$. If $1 < B$, then $x_1 = \frac{A}{x_0}$, $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0 \min\left\{1, \frac{1}{B}\right\} = \frac{x_0}{B}$, $x_3 = \max\left\{\frac{AB}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{x_0}\max\left\{1, \frac{1}{B}\right\} = \frac{AB}{x_0}$ and $x_4 = \max\left\{\frac{x_0}{B}, \frac{x_0}{B^2}\right\} = \frac{x_0}{B}\min\left\{1, \frac{1}{B}\right\} = \frac{x_0}{B^2}$. By induction we get $x_{2n} = \frac{x_0}{B^n}$ and $x_{2n-1} = \frac{AB^{n-1}}{x_0}$ for $1 \le n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B^2}, \frac{AB^2}{x_0}, ..., \frac{AB^{n-1}}{x_0}, \frac{x_0}{B^n}, ...)$$

If $B \in (0,1)$, then $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0 \min\left\{B, \frac{1}{B}\right\} = x_0$ and $x_3 = \max\left\{\frac{A}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{x_0}\max\left\{B, \frac{1}{B}\right\} = \frac{A}{Bx_0}$, $0 < \frac{A}{Bx_0}$, $x_4 = \max\left\{Bx_0, \frac{x_0}{B}\right\} = x_0 \min\left\{B, \frac{1}{B}\right\} = Bx_0$, $Bx_0 < 0$. By induction we get $x_{2n} = B^{n-1}x_0$ and $x_{2n-1} = \frac{A}{B^{n-1}x_0}$ for $1 \le n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{Bx_0}, Bx_0, ..., \frac{A}{B^{n-1}x_0}, B^{n-1}x_0, ...)$$

Also, it is easy to see that $x_{2n} < 0 < x_{2n-1}$ for $1 \leq n$, in this case $x_{-1}, x_0 < 0$.

(d), (e) Let $x_{-1} < 0 < x_0$ and $x_1 = \frac{A}{x_0}$, $\frac{A}{x_0} < 0$. If $1 < B$, then$x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0 \max\left\{1, \frac{1}{B}\right\} = x_0$, $x_3 = \max\left\{\frac{A}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{x_0}\min\left\{1, \frac{1}{B}\right\} = \frac{A}{Bx_0}$, $\frac{A}{Bx_0} < 0$ $x_4 = \max\left\{Bx_0, \frac{x_0}{B^2}\right\} = x_0 \max\left\{1, \frac{1}{B}\right\} = Bx_0$, $0 < Bx_0$. Hence by induction we get $x_{2n} = B^{n-1}x_0$, $0 < B^{n-1}x_0$ and $x_{2n-1} = \frac{A}{B^{n-1}x_0}$, $\frac{A}{B^{n-1}x_0} < 0$, for $1 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{Bx_0}, Bx_0, ..., \frac{A}{B^{n-1}x_0}, B^{n-1}x_0, ...)$$

If $B \epsilon (0,1)$, then $x_1 = \frac{A}{x_0}$, $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0 \max\left\{1, \frac{1}{B}\right\} = \frac{x_0}{B}$, $0 < \frac{x_0}{B}$ and $x_3 = \max\left\{\frac{AB}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{x_0}\min\left\{B, \frac{1}{B}\right\} = \frac{AB}{x_0}$, $x_4 = \max\left\{\frac{x_0}{B}, \frac{x_0}{B^2}\right\} = \frac{x_0}{B}\max\left\{1, \frac{1}{B}\right\} = \frac{x_0}{B^2}$. By induction we get $x_{2n} = \frac{x_0}{B^n}$, $0 < \frac{x_0}{B^n}$ and $x_{2n-1} = \frac{AB^{n-1}}{x_0}$, $\frac{AB^{n-1}}{x_0} < 0$ for $1 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B^2}, \frac{AB^2}{x_0}, ..., \frac{AB^{n-1}}{x_0}, \frac{x_0}{B^n}, ...)$$

(f), (g), (h) Let $x_{-1} < 0 < x_0$ and $x_1 = \frac{x_{-1}}{B}$, then we have $x_{2n-1} < 0 < x_{2n}$, $0 \leq n$. If $1 < B$, then

$x_2 = \max\left\{\frac{A}{x_1}, \frac{x_0}{B}\right\} = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{AB}{x_{-1}}$, $0 < \frac{AB}{x_{-1}}$, $x_3 = \max\left\{\frac{x_{-1}}{B}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B}\min\left\{1, \frac{1}{B}\right\} = \frac{x_{-1}}{B^2}$, $\frac{x_{-1}}{B^2} < 0$. Hence by induction we get $x_{2n} = \frac{AB^n}{x_{-1}}$ and $x_{2n-1} = \frac{x_{-1}}{B^n}$ for $1 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B^2}, \frac{AB^2}{x_{-1}}, ..., \frac{x_{-1}}{B^n}, \frac{AB^n}{x_{-1}}, ...)$$

If $B \epsilon (0,1)$ and $\frac{AB}{x_{-1}} < \frac{x_0}{B}$, then $x_1 = \frac{x_{-1}}{B}$, $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{x_0}{B}$, $0 < \frac{x_0}{B}$ and it follows $x_3 = \max\left\{\frac{AB}{x_0}, \frac{x_{-1}}{B^2}\right\} = \frac{AB}{x_0}$, $\frac{x_{-1}}{B^2} < \frac{x_{-1}}{B} < \frac{AB}{x_0}$ and $x_4 = \max\left\{\frac{x_0}{B}, \frac{x_0}{B^2}\right\} = \frac{x_0}{B}\max\left\{1, \frac{1}{B}\right\} = \frac{x_0}{B^2}$. By induction we get $x_{2n} = \frac{x_0}{B^n}$ and $x_{2n-1} = \frac{AB^{n-1}}{x_0}$ for $2 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B^2}, \frac{AB^2}{x_0}, ..., \frac{AB^{n-1}}{x_0}, \frac{x_0}{B^n}, ...)$$

If $B \epsilon (0,1)$ and $\frac{x_0}{B} < \frac{AB}{x_{-1}}$, then $x_1 = \frac{x_{-1}}{B}$, $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{AB}{x_{-1}}$ and $x_3 = \max\left\{\frac{x_{-1}}{B}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B}\min\left\{1, \frac{1}{B}\right\} = \frac{x_{-1}}{B}$, it follows $x_4 =$

$\max\left\{\frac{AB}{x_{-1}}, \frac{A}{x_{-1}}\right\} = \frac{A}{x_{-1}}$, $x_5 = \max\left\{x_{-1}, \frac{x_{-1}}{B^2}\right\} = x_{-1}$, $x_6 = \max\left\{\frac{A}{x_{-1}}, \frac{A}{Bx_{-1}}\right\} = \frac{A}{Bx_{-1}}$. By induction we get $x_{2n} = \frac{A}{B^{n-2}x_{-1}}$ and $x_{2n+1} = B^{n-2}x_{-1}$ for $1 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B}, \frac{A}{x_{-1}}, x_{-1}, \frac{A}{Bx_{-1}}, Bx_{-1}, ..., \frac{A}{B^{n-2}x_{-1}}, B^{n-2}x_{-1}, ...)$$

(i), (j) Let $x_0 < 0 < x_{-1}$ and $x_1 = \frac{A}{x_0}$, then $x_{2n} < 0 < x_{2n-1}$ for $0 \leq n$. If $1 < B$, then

$x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0 \min\left\{1, \frac{1}{B}\right\} = \frac{x_0}{B}$, $x_3 = \max\left\{\frac{AB}{x_0}, \frac{A}{Bx_0}\right\} = \frac{AB}{x_0}$, $x_4 = \max\left\{\frac{x_0}{B}, \frac{x_0}{B^2}\right\} = \frac{x_0}{B^2}$.Hence by induction we get $x_{2n} = \frac{x_0}{B^n}$ and $x_{2n-1} = \frac{AB^{n-1}}{x_0}$ for $1 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{AB}{x_0}, \frac{x_0}{B^2}, \frac{AB^2}{x_0}, ..., \frac{AB^{n-1}}{x_0}, \frac{x_0}{B^n}, ...)$$

If $B \epsilon (0,1)$, then $x_1 = \frac{A}{x_0}$, $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0$ and $x_3 = \max\left\{\frac{A}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{Bx_0}$, $x_4 = \max\left\{Bx_0, \frac{x_0}{B}\right\} = Bx_0$. By induction we get $x_{2n} = B^{n-1}x_0$ and $x_{2n-1} = \frac{A}{B^{n-1}x_0}$ for $1 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{Bx_0}, Bx_0, ..., \frac{A}{B^{n-1}x_0}, B^{n-1}x_0, ...)$$

(k), (l), (m) Let first $x_0 < 0 < x_{-1}$, $x_1 = \frac{x_{-1}}{B}$ and $1 < B$, If $\frac{x_0}{B} < \frac{AB}{x_{-1}}$, then $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{AB}{x_{-1}}$, and it follows $x_3 = \max\left\{\frac{x_{-1}}{B}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B}$, $x_4 = \max\left\{\frac{AB}{x_{-1}}, \frac{A}{x_{-1}}\right\} = \frac{A}{x_{-1}} \min\{B, 1\} = \frac{A}{x_{-1}}$. Hence by induction we get $x_{2n} = \frac{A}{B^{n-2}x_{-1}}$ and $x_{2n+1} = B^{n-2}x_{-1}$ for $1 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B}, \frac{A}{x_{-1}}, x_{-1}, \frac{A}{Bx_{-1}}, Bx_{-1}, ..., \frac{A}{B^{n-2}x_{-1}}, B^{n-2}x_{-1}, ...)$$

If $\frac{AB}{x_{-1}} < \frac{x_0}{B}$, then $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{x_0}{B}$, $\frac{x_{-1}}{B^2} < \frac{x_{-1}}{B} < \frac{AB}{x_0}$ and it follows $x_3 = \max\left\{\frac{AB}{x_0}, \frac{x_{-1}}{B^2}\right\} = \frac{AB}{x_0}$, $x_4 = \max\left\{\frac{x_0}{B}, \frac{x_0}{B^2}\right\} = \frac{x_0}{B} \min\left\{1, \frac{1}{B}\right\} = \frac{x_0}{B^2}$. By induction we get $x_{2n} = \frac{x_0}{B^n}$ and $x_{2n+1} = \frac{AB^n}{x_0}$ for $1 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{AB}{x}, \frac{x_0}{B^2}, \frac{AB^2}{x_0}, ..., \frac{x_0}{B^n}, \frac{AB^n}{x_0}, ...)$$

Finally, Let $x_0 < 0 < x_{-1}$, $x_1 = \frac{x_{-1}}{B}$ and $0 < B < 1$ then $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{AB}{x_{-1}}$ and $x_3 = \max\left\{\frac{x_{-1}}{B}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B}\max\left\{1, \frac{1}{B}\right\} = \frac{x_{-1}}{B^2}$, it follows $x_4 = \max\left\{\frac{AB^2}{x_{-1}}, \frac{A}{x_{-1}}\right\} = \frac{A}{x_{-1}}\min\left\{B^2, 1\right\} = \frac{AB^2}{x_{-1}}$. By induction we get $x_{2n} = \frac{AB^n}{x_{-1}}$ and $x_{2n-1} = \frac{x_{-1}}{B^n}$ for $1 \leq n$, that is

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B^2}, \frac{AB^2}{x_{-1}}, ..., \frac{x_{-1}}{B^n}, \frac{AB^n}{x_{-1}}, ...)$$

Also, it is easy to see that $x_{2n} < 0 < x_{2n-1}$, $0 \leq n$. The proof is completed.

2.3. Case III $A, B < 0$.In this section we consider the behaviour of the solutions of (1) in the case $A, B < 0$.

Lemma 1 *Consider the difference equation*

$$y_{n+1} = \left\{ \begin{array}{ll} \min\left\{A, \frac{y_n}{B}\right\} & n \equiv 0 \pmod 3 \\ \max\left\{A, \frac{y_n}{B}\right\} & n \neq 0 \pmod 3 \end{array} \right\} \tag{3}$$

where $y_0 \ \epsilon\ (-\infty, 0)$. *Then evey solution of (3) is eventually three periodic. Morever, the following statements are true;*

a) If $B \ \epsilon\ (-1, 0)$, *then (for* $3 \leq n$),

$$y_n = \left\{ \begin{array}{ll} \frac{A}{B} & n \equiv 1 \pmod 3 \\ A & n \neq 1 \pmod 3 \end{array} \right\}$$

b) If $B \leq -1$, *then (for* $3 \leq n$),

$$y_n = \left\{ \begin{array}{ll} A & n \equiv 0 \pmod 3 \\ \frac{A}{B} & n \equiv 1 \pmod 3 \\ \frac{A}{B^2} & n \equiv 2 \pmod 3 \end{array} \right\}$$

Proof. (a) Let $B \ \epsilon\ (-1, 0)$, then $y_1 = \max\left\{A, \frac{y_0}{B}\right\} = \frac{y_0}{B}$, $A < 0 < \frac{y_0}{B}$. If $A < \frac{y_0}{B^2}$, then $y_2 = \max\left\{A, \frac{y_0}{B^2}\right\} = \frac{y_0}{B^2}$ and $y_3 = \min\left\{A, \frac{y_0}{B^3}\right\} = A$. If $\frac{y_0}{B^2} < A$, then $y_2 = \max\left\{A, \frac{A}{B}\right\} = A$ and $y_3 = \min\left\{A, \frac{A}{B}\right\} = A$. It is easy to see that $y_3 = A$ (certainly) it follows $y_4 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$, $y_5 = \max\left\{A, \frac{A}{B^2}\right\} = A \min\left\{1, \frac{1}{B^2}\right\} = A$ and $y_6 = \min\left\{A, \frac{A}{B}\right\} = A = y_3$. Hence by induction we obtain the each solution of (3) is eventually three periodic and that is

$$y_n = \left\{ \begin{array}{ll} \frac{A}{B} & n \equiv 1 \pmod 3 \\ A & n \neq 1 \pmod 3 \end{array} \right\}, \quad for\ n \leq 3$$

(b) Let $B \leq -1$, then $y_1 = \max\left\{A, \frac{y_0}{B}\right\} = \frac{y_0}{B}$. If $\frac{y_0}{B^2} < A$, then $y_2 = \max\left\{A, \frac{y_0}{B^2}\right\} = A$ and $y_3 = \min\left\{A, \frac{A}{B}\right\} = A\max\left\{1, \frac{1}{B}\right\} = A$. If $A < \frac{y_0}{B^2}$, then $y_2 = \max\left\{A, \frac{y_0}{B^2}\right\} = \frac{y_0}{B^2}$ and $y_3 = \min\left\{A, \frac{y_0}{B^3}\right\} = A$. It is easy to see that $y_3 = A$ (certainly) it follows $y_4 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$, $y_5 = \max\left\{A, \frac{A}{B^2}\right\} = A$

$\min\left\{1, \frac{1}{B^2}\right\} = \frac{A}{B^2}$. By induction we obtain the each solution of (3) is eventually three periodic and that is

$$y_n = \left\{ \begin{array}{ll} A & n \equiv 0 \pmod 3 \\ \frac{A}{B} & n \equiv 1 \pmod 3 \\ \frac{A}{B^2} & n \equiv 2 \pmod 3 \end{array} \right\}, \quad for\ n \leq 3$$

Lemma 2 *Consider the difference equation*

$$(4) \qquad y_{n+1} = \left\{ \begin{array}{ll} \min\left\{A, \frac{y_n}{B}\right\} & n \equiv 1 \pmod 3 \\ \max\left\{A, \frac{y_n}{B}\right\} & n \neq 1 \pmod 3 \end{array} \right\}$$

where $y_0 \epsilon R - \{0\}$. Then every solution of (4) is eventually three periodic. Morever, the following statements are true;

a) If $B \leq -1$, then

$$y_n = \left\{ \begin{array}{ll} A & n \equiv 1 \pmod 3 \\ \frac{A}{B} & n \equiv 2 \pmod 3 \\ \frac{A}{B^2} & n \equiv 0 \pmod 3 \end{array} \right\}, \text{for } 4 \leq n$$

b) If $B \epsilon (-1, 0)$, then

$$y_n = \left\{ \begin{array}{ll} \frac{A}{B} & n \equiv 2 \pmod 3 \\ A & n \neq 2 \pmod 3 \end{array} \right\}, \text{ for } 4 \leq n$$

Proof. (a) Let $B \epsilon (-\infty, -1]$ and $0 < y_0$. If $A < \frac{y_0}{B}$, then $y_1 = \min\left\{A, \frac{y_0}{B}\right\} = A$, $y_2 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$, $y_3 = \max\left\{A, \frac{A}{B^2}\right\} = \frac{A}{B^2}$ and $y_4 = \min\left\{A, \frac{A}{B^3}\right\} = A\max\left\{1, \frac{1}{B^3}\right\} = A = y_1$. By induction we get

$$y_n = \left\{ \begin{array}{ll} A & n \equiv 1 \pmod 3 \\ \frac{A}{B} & n \equiv 2 \pmod 3 \\ \frac{A}{B^2} & n \equiv 0 \pmod 3 \end{array} \right\}, \text{for } 1 \leq n$$

Let $y_1 = \frac{y_0}{B}$, then $y_2 = \max\left\{A, \frac{y_0}{B^2}\right\} = \frac{y_0}{B^2}$. If $y_3 = \max\left\{A, \frac{y_0}{B^3}\right\} = \frac{y_0}{B^3}$, then $y_4 = \min\left\{A, \frac{y_0}{B^4}\right\} = A$. If $y_3 = \max\left\{A, \frac{y_0}{B^3}\right\} = A$, then $y_4 = \min\left\{A, \frac{A}{B}\right\} = A$. We have $y_4 = A$. It follows $y_5 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$, $y_6 = \max\left\{A, \frac{A}{B^2}\right\} = \frac{A}{B^2}$ and $y_7 = \min\left\{A, \frac{A}{B^3}\right\} = A = y_4$. Hence, by induction, we obtain

$$y_n = \left\{ \begin{array}{ll} A & n \equiv 1 \pmod 3 \\ \frac{A}{B} & n \equiv 2 \pmod 3 \\ \frac{A}{B^2} & n \equiv 0 \pmod 3 \end{array} \right\}, \text{for } 4 \leq n$$

Now, Let $B \epsilon (-\infty, -1]$ and $y_0 < 0$. Then $y_1 = \min\left\{A, \frac{y_0}{B}\right\} = A$, $A < 0 < \frac{y_0}{B}$ and $y_2 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$, $y_3 = \max\left\{A, \frac{A}{B^2}\right\} = A\min\left\{1, \frac{1}{B^2}\right\} = \frac{A}{B^2}$ and $y_4 = \min\left\{A, \frac{A}{B^3}\right\} = A\max\left\{1, \frac{1}{B^3}\right\} = A = y_1$. By induction, we get

$$y_n = \left\{ \begin{array}{ll} A & n \equiv 1 \pmod 3 \\ \frac{A}{B} & n \equiv 2 \pmod 3 \\ \frac{A}{B^2} & n \equiv 0 \pmod 3 \end{array} \right\}, \text{for } 1 \leq n$$

(b) Let $B \epsilon (-1,0)$, $0 < y_0$ and $y_1 = A$, then $y_2 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$, $y_3 = \max\left\{A, \frac{A}{B^2}\right\} = A$, $y_4 = \min\left\{A, \frac{A}{B}\right\} = A = y_1$. Hence by induction we see that each solution of Eq.(4) is three periodic and

$$y_n = \left\{ \begin{array}{cc} \frac{A}{B} & n \equiv 2 \pmod 3 \\ A & n \neq 2 \pmod 3 \end{array} \right\}, \text{ for } 1 \leq n$$

Let $y_1 = \frac{y_0}{B}$, $\frac{y_0}{B} < A$, then $y_2 = \max\left\{A, \frac{y_0}{B^2}\right\} = \frac{y_0}{B^2}$. If $A < \frac{y_0}{B^3}$ or $\frac{y_0}{B^3} < A$, then by induction we have $y_4 = A$. It follows $y_5 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$, $y_6 = \max\left\{A, \frac{A}{B^2}\right\} = A$ and $y_7 = \min\left\{A, \frac{A}{B}\right\} = A = y_4$. By induction we obtain

$$y_n = \left\{ \begin{array}{cc} \frac{A}{B} & n \equiv 2 \pmod 3 \\ A & n \neq 2 \pmod 3 \end{array} \right\}, \text{ for } 4 \leq n$$

Thus it is eventually three periodic.

Finally, Let $y_0 < 0$. then $y_1 = \min\left\{A, \frac{y_0}{B}\right\} = A$ and it follows $y_2 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$, $y_3 = \max\left\{A, \frac{A}{B^2}\right\} = A$ and $y_4 = \min\left\{A, \frac{A}{B}\right\} = A = y_1$. By induction, we get

$$y_n = \left\{ \begin{array}{cc} \frac{A}{B} & n \equiv 2 \pmod 3 \\ A & n \neq 2 \pmod 3 \end{array} \right\}, \text{ for } 1 \leq n$$

Thus, the proof is completed.

Lemma 3 *Consider the difference equation*

$$w_{n+1} = \left\{ \begin{array}{cc} \min\left\{A, \frac{w_n}{B}\right\} & n \equiv 2 \pmod 3 \\ \max\left\{A, \frac{w_n}{B}\right\} & n \neq 2 \pmod 3 \end{array} \right\} \quad (5)$$

Then every solution of (5) is eventually three periodic. Morever the following statements are true for $w_0 > 0$;

(a) If $B \epsilon (-\infty, -1]$ *, then*

$$w_n = \left\{ \begin{array}{cc} \frac{A}{B} & n \equiv 0 \pmod 3 \\ \frac{A}{B^2} & n \equiv 1 \pmod 3 \\ A & n \equiv 2 \pmod 3 \end{array} \right\}, \text{for } 2 \leq n$$

(b) If $B \epsilon (-1, 0)$*, then*

$$w_n = \left\{ \begin{array}{cc} \frac{A}{B} & n \equiv 0 \pmod 3 \\ A & n \neq 0 \pmod 3 \end{array} \right\}, \text{ for } 2 \leq n$$

Proof. (a) Let $B \epsilon (-\infty, -1]$ and $0 < w_0$. If $w_1 = A$ or $w_1 = \frac{w_0}{B}$,then, certainly, $w_2 = \min\left\{A, \frac{A}{B}\right\} = \min\left\{A, \frac{y_0}{B^2}\right\} = A$, $w_3 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$ and $w_4 = \max\left\{A, \frac{A}{B^2}\right\} = \frac{A}{B^2}$. By induction we get

$$w_n = \left\{ \begin{array}{cc} \frac{A}{B} & n \equiv 0 \pmod 3 \\ \frac{A}{B^2} & n \equiv 1 \pmod 3 \\ A & n \equiv 2 \pmod 3 \end{array} \right\}, \text{for } 2 \leq n$$

It is eventually three periodic.

(b) Let $B \epsilon (-1,0)$, then $w_1 = \max\left\{A, \frac{y_0}{B}\right\} = A$ or $w_1 = \max\left\{A, \frac{y_0}{B}\right\} = \frac{y_0}{B}$, But certainly $w_2 = \min\left\{A, \frac{A}{B}\right\} = \min\left\{A, \frac{y_0}{B^2}\right\} = A$. Then $w_3 = \max\left\{A, \frac{A}{B}\right\} = \frac{A}{B}$ and $w_4 = \max\left\{A, \frac{A}{B^2}\right\} = A$ and $w_5 = \min\left\{A, \frac{A}{B}\right\} = A = w_2$. Hence by induction we obtain

$$w_n = \left\{ \begin{array}{ll} \frac{A}{B} & n \equiv 0 \pmod 3 \\ A & n \neq 0 \pmod 3 \end{array} \right\}, \text{ for } 2 \leq n$$

Also, It is easy to see that it is eventually three periodic. The proof is completed.

Theorem 3. *Consider (1). If* $A, B < 0$, *then every solution of (1) is eventually six periodic.*

Proof. (a) Let $0 < x_0, x_{-1}$, then we have $0 < x_{3n}$ and $x_{3n+1} < 0 < x_{3n+2}$ for $0 \leq n$.

We can multiply (1) by x_n and use the equality $w_n = x_n x_{n-1}$ to obtain (5). Since all conditions of Lemma 3 are satisfied, we see that in this case the sequence w_n is eventually three periodic. It means that each solution (x_n) of (1) is eventually six periodic in this case.

(b) Let $x_0 < 0$ and $x_{-1} \epsilon R - \{0\}$, then we easily write $x_{3n} < 0 < x_{3n+1}$, x_{3n+2} for $0 \leq n$.

We can multiply (1) by x_n and use the substitution $y_n = x_n x_{n-1}$, to obtain (4). Since all conditions of Lemma 2 are satisfied we see that in this case the sequence y_n is eventually three periodic. We can say that each solution (x_n) of (1) is eventually six periodic.

(c) Finally, let $x_{-1} < 0 < x_0$, then we have $0 < x_{3n}$ and $x_{3n+2} < 0 < x_{3n+1}$ for $0 \leq n$.

We can multiply (1) by x_n and use $y_n = x_n x_{n-1}$. Therefore we obtain (3). Since Lemma 1 is satisfied, in this case, evey solution (y_n) of (3) is eventually three periodic. It means that each solution (x_n) of (1) is eventually six periodic. The proof is completed.

2.4. Case IV $0 < A, B$. In this section we consider the behaviour of the solutions of (1) in the case $0 < A, B$. Prior to investigating the behaviour of the solutions of (1), we prove two auxiliary results.

Firstly, let $0 < x_{-1}, x_0$, then $0 < x_n$ for $-1 \leq n$, which is each solution of (1). We can multiply (1) by x_n use the change $y_n = x_n x_{n-1}$. We obtain the equation

$$y_{n+1} = \max\left\{A, \frac{y_n}{B}\right\}, \quad 0 \leq n \tag{6}$$

where $0 < A, B$ and $0 < y_0$.

Secondly, let $x_{-1}, x_0 < 0$, then $x_n < 0$ for $-1 \leq n$, which is each solution of (1). We can multiply (1) by x_n and use the equality $y_n = x_n x_{n-1}$. We obtain the equation

$$(7) \qquad y_{n+1} = \min\left\{A, \frac{y_n}{B}\right\}, \quad 0 \leq n$$

where $0 < A, B$ and $0 < y_0$.

Lemma 4 *Consider (6). Then the following statements are true;*

(a) Let $1 \leq B$, then each solution y_n of (6) is eventually constant.

(b) Let $B \ \epsilon \ (0,1)$, then each solution y_n of (6) is eventually satisfies the difference equation $y_{n+1} = \frac{y_n}{B}$.

Proof. (a) Let, $1 < B$. If $y_0 \ \epsilon \ (0, AB]$, then $y_1 = A$. Since $\frac{y_0}{B} \leq A$, it follows $\frac{y_1}{B} < A$ which implies $y_2 = A$. By induction we have $y_n = A$ for $1 \leq n$.

If $AB < y_0$, then $y_1 = \frac{y_0}{B}$. If $\frac{y_0}{B^2} \leq 1$, then $y_2 = A$ and consequently $y_n = A$ for $2 \leq n$. In contrary $y_2 = A^2 y_0$. Since $1 < B$, we have $B^n \to \infty$ as $n \to \infty$. Hence, there is a number $n_0 \epsilon N$ such that $\frac{y_0}{B^{n_0}} \leq A$ and $A < \frac{y_0}{B^{n_0 - 1}}$. It is easy to see that $y_n = A$ for $n_0 \leq n$, as desired.

If $B = 1$, then $y_0 \ \epsilon \ (0, A]$, we have $y_1 = A$ and consequently $y_n = A$ for $1 \leq n$.

If $A < y_0$, then $y_1 = y_0$, $1 < y_0$ and by induction $y_n = y_0$ for $0 \leq n$, that want to prove.

(b) If $y_0 \ \epsilon \ (0, AB]$, then $y_1 = A$. Further, $y_2 = \max\left\{A, \frac{y_1}{B}\right\} = \frac{y_1}{B}$, $A < \frac{y_1}{B} = \frac{A}{B}$. By induction we obtain $y_n \leq y_{n+1}$ for $1 \leq n$ which implies $y_{n+1} = \frac{y_n}{B}$ for $1 \leq n$.

If $AB < y_0$, then $y_1 = \frac{y_0}{B}$, $A < \frac{y_0}{B}$. From, this it follow, that $y_{n+1} = \frac{y_n}{B}$ for $0 \leq n$. The proof is completed.

The following lemma can be considered as a dual result of Lemma 4.

Lemma 5 *Consider (7) Then the following statements are true;*

(a) Let $1 < B$, then each solution y_n of (7) is eventually satisfies the difference equation $y_{n+1} = \frac{y_n}{B}$.

(b) Let $B \ \epsilon \ (0,1]$, then each solution y_n of (7) is eventually constant.

Proof. (a) If $AB < y_0$, then $y_1 = A$ and $y_2 = \min\left\{A, \frac{y_1}{B}\right\} = \frac{y_1}{B} = \frac{A}{B}$, $\frac{A}{B} < A$. Hence, by induction, we get $y_{n+1} = \frac{y_n}{B}$ for $1 \leq n$.

If $y_0 \ \epsilon \ (0, AB]$, then $y_1 = \frac{y_0}{B}$, $\frac{y_0}{B} \leq A$ and $y_2 = \min\left\{A, \frac{y_1}{B}\right\} = \frac{y_1}{B} = \frac{y_0}{B^2}$. Thus by induction $y_{n+1} = \frac{y_n}{B}$ for $0 \leq n$.

(b) Let $B \ \epsilon \ (0,1)$. If $AB < y_0$, then it follows $y_1 = A$ and $y_2 = \min\left\{A, \frac{A}{B}\right\} = A < \frac{A}{B}$. It is easy to see that $y_n = A$ for $1 \leq n$.

If $y_0 < AB$, then $y_1 = \frac{y_0}{B}$ and If $AB^2 \leq y_0$ then $y_2 = A$. Hence we get $y_n = A$ for $2 \leq n$. If $y_0 < AB^2$, then Since $B \epsilon (0,1)$, $B^n \to \infty$ as $n \to \infty$. Hence, There is an $m_0 \epsilon N$ such that $AB^{m_0} \leq y_0$ and $y_0 < AB^{m_0-1}$. For such chosen m_0 we have $y_{m_0} = A$,which implies $y_n = A$ for $m_0 \leq n$.

Finally, Let $B = 1$ and If $A \leq y_0$, then $y_1 = A$ and $y_2 = \min\{A, y_1\} = A = y_1$. Hence we get $y_n = A$ for $1 \leq n$. If $y_0 \epsilon (0, A)$, then $y_1 = y_0$, $y_0 < A$ and $y_2 = \min\{A, y_1\} = y_1 = y_0$. Hence we obtain that $y_n = y_0$ for $0 \leq n$. The proof is completed.

Theorem 4 *Consider (1), with* $0 < A, B$.

a) If $0 < x_0, x_{-1}$, $x_1 = \frac{A}{x_0}$ *and* $1 \leq B$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, ...)$$

b) If $0 < x_0, x_{-1}$, $x_1 = \frac{A}{x_0}$ *and* $0 < B < 1$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{A}{Bx_0}, \frac{x_0}{B^2}, ..., \frac{A}{B^{n-1}x_0}, \frac{x_0}{B^n}, ...)$$

c) If $0 < x_0, x_{-1}$, $x_1 = \frac{x_{-1}}{B}$ *and* $1 \leq B$, *then* (x_n) *is eventually two periodic.*

d) If $0 < x_0, x_{-1}$, $x_1 = \frac{x_{-1}}{B}$ *and* $0 < B < 1$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{x_{-1}}{B^2}, \frac{x_0}{B^2}, ..., \frac{x_{-1}}{B^n}, \frac{x_0}{B^n}, ...)$$

e) If $x_{-1}, x_0 < 0$, $x_1 = \frac{A}{x_0}$ *and* $1 < B$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{A}{Bx_0}, \frac{x_0}{B^2}, ..., \frac{A}{B^{n-1}x_0}, \frac{x_0}{B^n}, ...)$$

f) If $x_{-1}, x_0 < 0$, $x_1 = \frac{A}{x_0}$ *and* $0 < B \leq 1$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, ...)$$

g) If $x_{-1}, x_0 < 0$, $x_1 = \frac{x_{-1}}{B}$ *and* $1 < B$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{x_{-1}}{B^2}, \frac{x_0}{B^2}, ..., \frac{x_{-1}}{B^n}, \frac{x_0}{B^n}, ...)$$

h) If $x_{-1}, x_0 < 0$, $x_1 = \frac{x_{-1}}{B}$ *and* $0 < B \leq 1$, *then* (x_n) *is eventually two periodic.*

i) If $x_{-1} < 0 < x_0$ *and* $1 \leq B$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, ...)$$

j) If $x_{-1} < 0 < x_0$ *and* $0 < B < 1$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{A}{Bx_0}, \frac{x_0}{B^2}, ..., \frac{A}{B^{n-1}x_0}, \frac{x_0}{B^n}, ...)$$

k) If $x_0 < 0 < x_{-1}$ *and* $1 \leq B$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, ...)$$

l) If $x_0 < 0 < x_{-1}$ *and* $0 < B < 1$, *then*

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B^2}, \frac{A}{x_{-1}}, \frac{x_{-1}}{B^3}, \frac{A}{Bx_{-1}}, ..., \frac{x_{-1}}{B^n}, \frac{A}{B^{n-2}x_{-1}}, ...)$$

Proof. (a), (b) Let $0 < x_{-1}, x_0$ and $x_1 = \frac{A}{x_0}$. If $1 \leq B$, then $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0 \max\left\{1, \frac{1}{B}\right\} = x_0$, $0 < x_0$ and $x_3 = \max\left\{\frac{A}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{x_0}\max\left\{1, \frac{1}{B}\right\} = \frac{A}{x_0}$. By induction, we obtain $x_{2n} = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0$ and $x_{2n-1} = \max\left\{\frac{A}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{x_0}$ for $1 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, x_0, \frac{A}{x_0}, ...)$$

If $0 < B < 1$ and $x_1 = \frac{A}{x_0}$, then $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = \frac{x_0}{B}$, $0 < \frac{x_0}{B}$, $x_3 = \max\left\{\frac{AB}{x_0}, \frac{A}{Bx_0}\right\} = \frac{A}{x_0}\max\left\{B, \frac{1}{B}\right\} = \frac{AB}{x_0}$ and $x_4 = \max\left\{\frac{x_0}{B}, \frac{x_0}{B^2}\right\} = \frac{x_0}{B^2}$. By induction we get $x_{2n} = \frac{x_0}{B^n}$ and $x_{2n-1} = \frac{A}{B^{n-1}x_0}$ for $1 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{A}{x_0}, \frac{x_0}{B}, \frac{A}{Bx_0}, \frac{x_0}{B^2}, ..., \frac{A}{B^{n-1}x_0}, \frac{x_0}{B^n}, ...)$$

(c), (d) Let $0 < x_{-1}, x_0$, then $x_1 = \frac{x_{-1}}{B}$. The case when $\frac{A}{x_0} < \frac{x_{-1}}{B}$ and $1 \leq B$ is more complicated. Because $0 < x_n$ for $-1 \leq n$, we can multiply (1) by (x_n) and use the substitution $y_n = x_n x_{n-1}$, we obtain (6). Since all conditions of Lemma 4 are satisfied we see that the sequence (y_n) is eventually constant. It means that each solution (x_n) of (1), in the case, is eventually two periodic. If $0 < B < 1$ and $x_1 = \frac{x_{-1}}{B}$, then $\frac{AB}{x_{-1}} < x_0$, $\frac{AB}{x_{-1}} < x_0 < \frac{x_0}{B}$ and $\frac{AB}{x_0} < \frac{x_{-1}}{B} < \frac{x_{-1}}{B^2}$. Hence $x_2 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B}\right\} = \frac{x_0}{B}$, and $x_3 = \max\left\{\frac{AB}{x_0}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B^2}$. By induction, we get $x_{2n} = \frac{x_0}{B^n}$ and $x_{2n-1} = \frac{x_{-1}}{B^n}$ for $1 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{x_0}{B}, \frac{x_{-1}}{B^2}, \frac{x_0}{B^2}, ..., \frac{x_{-1}}{B^n}, \frac{x_0}{B^n}, ...)$$

(e), (f) Let $x_{-1}, x_0 < 0$, then $x_n < 0$ for $-1 \leq n$. If $1 < B$ and $x_1 = \frac{A}{x_0}$, then $x_2 = \max\left\{x_0, \frac{x_0}{B}\right\} = x_0 \min\left\{1, \frac{1}{B}\right\} = \frac{x_0}{B}$ and $x_3 = \max\left\{\frac{AB}{x_0}, \frac{A}{Bx_0}\right\} =$

$\frac{A}{x_0}\min\left\{B,\frac{1}{B}\right\}=\frac{A}{Bx_0}$ and $x_4=\max\left\{Bx_0,\frac{x_0}{B^2}\right\}=x_0\min\left\{B,\frac{1}{B^2}\right\}=\frac{x_0}{B^2}$, $\frac{x_0}{B^2}<0$. By induction, we get $x_{2n}=\frac{x_0}{B^n}$ and $x_{2n-1}=\frac{A}{B^{n-1}x_0}$ for $1\leq n$, that is,

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},\frac{x_0}{B},\frac{A}{Bx_0},\frac{x_0}{B^2},...,\frac{A}{B^{n-1}x_0},\frac{x_0}{B^n},...)$$

If $0<B\leq 1$ and $x_1=\frac{A}{x_0}$, then $x_2=\max\left\{x_0,\frac{x_0}{B}\right\}=x_0$, $\frac{x_0}{B}<x_0$ and $x_3=\max\left\{\frac{A}{x_0},\frac{A}{Bx_0}\right\}=\frac{A}{x_0}\min\left\{1,\frac{1}{B}\right\}=\frac{A}{x_0}$, Using induction we get $x_{n+1}=\frac{A}{x_n}$ for $1\leq n$. That is

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},x_0,\frac{A}{x_0},...)$$

(g), (h) Let $x_{-1},x_0<0$, then $x_n<0$ for $-1\leq n$. If $1<B$ and $x_1=\frac{x_{-1}}{B}$, then $x_2=\max\left\{\frac{AB}{x_{-1}},\frac{x_0}{B}\right\}=\frac{x_0}{B}$ and $x_3=\max\left\{\frac{AB}{x_0},\frac{x_{-1}}{B^2}\right\}=\frac{x_{-1}}{B^2}$. By induction we get $x_{2n}=\frac{x_0}{B^n}$ and $x_{2n-1}=\frac{x_{-1}}{B^n}$ for $1\leq n$, that is

$$(x_n)=(x_{-1},x_0,\frac{x_{-1}}{B},\frac{x_0}{B},\frac{x_{-1}}{B^2},\frac{x_0}{B^2},...,\frac{x_{-1}}{B^n},\frac{x_0}{B^n},...)$$

If $x_{-1},x_0<0$, $x_1=\frac{x_{-1}}{B}$ and $0<B\leq 1$, then since $x_n<0$ for $-1\leq n$, we can multiply (1) by (x_n) and use the change $y_n=x_nx_{n-1}$, we obtain that the sequence (y_n) satisfies (7) and $0<y_n$ for $0\leq n$. Since $0<B<1$ by Lemma 2 we obtain that (y_n) is eventually constant. which implies that (x_n) is eventually two periodic.

(i), (j) Let $x_{-1}<0<x_0$, then $x_1=\max\left\{\frac{A}{x_0},\frac{x_{-1}}{B}\right\}=\frac{A}{x_0}$, $\frac{x_{-1}}{B}<0<\frac{A}{x_0}$.If $1\leq B$, then $x_2=\max\left\{x_0,\frac{x_0}{B}\right\}=x_0\max\left\{1,\frac{1}{B}\right\}=x_0$ and $x_3=\max\left\{\frac{A}{x_0},\frac{A}{Bx_0}\right\}=\frac{A}{x_0}\max\left\{1,\frac{1}{B}\right\}=\frac{A}{x_0}=x_1$. Hence, by induction,we write $0<x_n$ and $x_{n+1}=\frac{A}{x_n}$ for $0\leq n$.That is

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},x_0,\frac{A}{x_0},...)$$

If $0<B<1$, then $x_1=\frac{A}{x_0}$and $x_2=\max\left\{x_0,\frac{x_0}{B}\right\}=\frac{x_0}{B}$ and $x_3=\max\left\{\frac{AB}{x_0},\frac{A}{Bx_0}\right\}=\frac{A}{x_0}\max\left\{B,\frac{1}{B}\right\}=\frac{A}{Bx_0}$ and $x_4=\max\left\{Bx_0,\frac{x_0}{B^2}\right\}=\frac{x_0}{B^2}$. Hence, by induction, we get $x_{2n}=\frac{x_0}{B^n}$ and $x_{2n-1}=\frac{A}{B^{n-1}x_0}$ for $1\leq n$, that is,

$$(x_n)=(x_{-1},x_0,\frac{A}{x_0},\frac{x_0}{B},\frac{A}{Bx_0},\frac{x_0}{B^2},...,\frac{A}{B^{n-1}x_0},\frac{x_0}{B^n},...)$$

(k), (l) Let $x_0<0<x_{-1}$, then $0<x_n$ for $1\leq n$. Also $x_1=\max\left\{\frac{A}{x_0},\frac{x_{-1}}{B}\right\}=\frac{x_{-1}}{B}$, $\frac{A}{x_0}<0<\frac{x_{-1}}{B}$ and $x_2=\max\left\{\frac{AB}{x_{-1}},\frac{x_0}{B}\right\}=\frac{AB}{x_{-1}}$, $\frac{x_0}{B}<0<\frac{AB}{x_{-1}}$.

If $1 \leq B$, then $x_3 = \max\left\{\frac{A}{x_2}, \frac{x_1}{B}\right\} = \max\left\{\frac{x_{-1}}{B}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B}\max\left\{1, \frac{1}{B}\right\} = \frac{x_{-1}}{B} = x_1$ and $x_4 = \max\left\{\frac{AB}{x_{-1}}, \frac{x_0}{B^2}\right\} = \frac{AB}{x_{-1}}$. Hence, by induction, we get $x_{2n} = \frac{AB}{x_{-1}}$ and $x_{2n-1} = \frac{x_{-1}}{B}$ for $1 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, ...)$$

If $0 < B < 1$, then $x_1 = \frac{x_{-1}}{B}$ and $x_2 = \frac{AB}{x_{-1}}$ and $x_3 = \max\left\{\frac{x_{-1}}{B}, \frac{x_{-1}}{B^2}\right\} = \frac{x_{-1}}{B}\max\left\{1, \frac{1}{B}\right\} = \frac{x_{-1}}{B^2}$ and $x_4 = \max\left\{\frac{AB^2}{x_{-1}}, \frac{A}{x_{-1}}\right\} = \frac{A}{x_{-1}}\max\left\{B^2, 1\right\} = \frac{A}{x_{-1}}$. Hence, by induction, we get $x_{2n} = \frac{A}{B^{n-2}x_{-1}}$ and $x_{2n-1} = \frac{x_{-1}}{B^n}$ for $1 \leq n$, that is,

$$(x_n) = (x_{-1}, x_0, \frac{x_{-1}}{B}, \frac{AB}{x_{-1}}, \frac{x_{-1}}{B^2}, \frac{A}{x_{-1}}, \frac{x_{-1}}{B^3}, \frac{A}{Bx_{-1}}, ..., \frac{x_{-1}}{B^n}, \frac{A}{B^{n-2}x_{-1}}, ...)$$

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