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On The Bounds of Spectral Norms of Some Special Matrices

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Summary. In this study, we have found bounds for the spectral norms of A, $A^{\circ(-1)}$, $A^{\circ(1/2)}$ and B using two different methods where $A = [1/(ij)]_{i,j=1}^{n}$ and $B = [\sqrt{ij}/(i+j)]_{i,j=1}^{n}$.

Key words: Hadamard inverse, Hadamard square root, norm, spectral norm.

1. Introduction

Let $A = (a_{ij})$ be $n \times n$ real matrix. The Hadamard inverse of the Matrix A is defined as

$$A^{\circ(-1)} = \left(\frac{1}{a_{ij}}\right)$$

such that $a_{ij} \neq 0$ for $1 \leqslant i,\, j \leqslant n$.

Let $A = (a_{ij})$ be $n \times n$ matrix. The Hadamard square root of the Matrix A is defined as

$$A^{\circ(1/2)} = \left(a_{ij}^{1/2}\right)$$

such that $a_{ij} \geqslant 0$ for $1 \leqslant i,\, j \leqslant n$.

$\gamma = 0.5772156649$ is known as Euler constant.

Let $A = (a_{ij})$ be $n \times n$ real matrix. The well known Euclidean norm of the matrix A is

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2} = \left(\sum_{i}^{n} \sigma_i^2(A)\right)^{1/2}$$

and also the spectral norm of the matrix A is

$$\|A\|_2 = \sqrt{\max_{1\leqslant\, i\leqslant\, n} |\lambda_i|}$$

where λ_i is eigenvalue of $A^H A$ and A^H is conjugate transpose of the matrix A. The folloeing inequality between $\|A\|_F$ and $\|A\|_2$ norms is holds [1]:

$$\||A\||_2 \leqslant \|A\|_F \leqslant \sqrt{n}\||A\||_2. \tag{1}$$

A function Ψ is called psi (or digamma) function if

$$\Psi(x) = \frac{d}{dx}\{\log[\Gamma(x)]\}$$

where

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

It is called polygamma function the n th derivatives of psi function [2] i.e.

$$\Psi(n, x) = \frac{d^n}{dx^n} Psi(x) = \frac{d^n}{dx^n}\left[\frac{d}{dx}\ln\left[\,\Gamma(x)\,\right]\right]$$

where if $n = 0$ then $\Psi(0, x) = Psi(x) = \frac{d}{dx}\{\ln\left[\Gamma(x)\right]\}$. On the other hand, if $a > 0$, b is any number and n is positive integer, then

$$\lim_{n\to\infty} \Psi(a,\, n + b) = 0.$$

Theorem 1 *[3] Let A, B, $C \in M_{m,n}$. If $A = B \circ C$ then*

$$\||A\||_2 \leqslant r_1(B)\, c_1(C) \tag{2}$$

such that

$$c_1(A) \equiv \max_j \sqrt{\sum_i |a_{ij}|^2} = \max_j \|[a_{ij}]_{i=1}^m\|_2$$

and

$$r_1(A) \equiv \max_i \sqrt{\sum_j |a_{ij}|^2} = \max_i \left\|[a_{ij}]_{j=1}^m\right\|_2.$$

2. The Bounds for Spectral Norms of Some Special Matrices

Theorem 2 *Let $A = [1/(ij)]_{i,j=1}^n$ be a matrix. Then*

i) $\frac{1}{\sqrt{n}}\left(\frac{\pi^2}{6}-\Psi(1,\,n+1)\right)\leqslant |||A|||_2 \leqslant \frac{\pi^2}{6}$

ii) $\frac{1}{\sqrt{n}}\left(\Psi(n+1)+\gamma\right)\leqslant \left|\left|\left|A^{\circ(1/2)}\right|\right|\right|_2 \leqslant \Psi(n+1)\,+\,\gamma$

iii) $\frac{\sqrt{n}\,(n+1)\,(2n+1)}{6}\leqslant \left|\left|\left|A^{\circ(-1)}\right|\right|\right|_2 \leqslant \frac{n\,(n+1)\,(2n+1)}{6}$

hold.

Proof. i) Let us write the matrix A clearly

$$A=\begin{bmatrix} \frac{1}{1.1} & \frac{1}{1.2} & \cdot & \cdot & \cdot & \frac{1}{1.n} \\ \frac{1}{2.1} & \frac{1}{2.2} & \cdot & \cdot & \cdot & \frac{1}{2.n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \frac{1}{n.1} & \frac{1}{n.2} & \cdot & \cdot & \cdot & \frac{1}{n.n} \end{bmatrix}.$$

Then the Euclidean norm of A is

$$\begin{aligned}
\|A\|_F^2 &= \left[\left(\tfrac{1}{1.1}\right)^2+\left(\tfrac{1}{1.2}\right)^2+...+\left(\tfrac{1}{1.n}\right)^2\right]+\left[\left(\tfrac{1}{2.1}\right)^2+\left(\tfrac{1}{2.2}\right)^2+...+\left(\tfrac{1}{2.n}\right)^2\right] \\
&\quad +\ ...\ +\left[\left(\tfrac{1}{n.1}\right)^2+\left(\tfrac{1}{n.2}\right)^2+\ ...\ +\left(\tfrac{1}{n.n}\right)^2\right] \\
&= \sum_{k=1}^{n}\left(\tfrac{1}{1.k}\right)^2+\sum_{k=1}^{n}\left(\tfrac{1}{2.k}\right)^2+\ ...\ +\sum_{k=1}^{n}\left(\tfrac{1}{n.k}\right)^2 \\
&= \tfrac{1}{1^2}\sum_{k=1}^{n}\tfrac{1}{k^2}+\tfrac{1}{2^2}\sum_{k=1}^{n}\tfrac{1}{k^2}+\ ...\ +\tfrac{1}{n^2}\sum_{k=1}^{n}\tfrac{1}{k^2} \\
&= \left(\sum_{k=1}^{n}\tfrac{1}{k^2}\right)\left(\tfrac{1}{1^2}+\tfrac{1}{2^2}+\ ...\ +\tfrac{1}{n^2}\right)=\left(\sum_{k=1}^{n}\tfrac{1}{k^2}\right)\left(\sum_{k=1}^{n}\tfrac{1}{k^2}\right) \\
&= \left(\sum_{k=1}^{n}\tfrac{1}{k^2}\right)^2.
\end{aligned}$$

If we evaluate $\left(\sum_{k=1}^{n}(1/k^2)\right)^2$, then we have

$$\left(\sum_{k=1}^{n}(1/k^2)\right)^2=\left(\frac{\pi^2}{6}-\Psi(1,\,n+1)\right)^2.$$

Hence

$$\|A\|_F=\sum_{k=1}^{n}(1/k^2)=\frac{\pi^2}{6}-\Psi(1,\ n+1). \tag{3}$$

The limit value for the right side of (3) as $n\to\infty$ is

(4) $$\lim_{n\to\infty}\sum\nolimits_{k=1}^{n}(1/k^2)=\frac{\pi^2}{6}.$$

From (1)

$$\frac{1}{\sqrt{n}}\left(\frac{\pi^2}{6}-\Psi(1\,,\,n+1\,)\right)\leqslant \||A\||_2\leqslant\frac{\pi^2}{6}$$

is obtained.

ii) If we write the matrix $A^{\circ(1/2)}$, then we have

$$A=\begin{bmatrix}\sqrt{\frac{1}{1.1}} & \sqrt{\frac{1}{1.2}} & . & . & . & \sqrt{\frac{1}{1.n}}\\ \sqrt{\frac{1}{2.1}} & \sqrt{\frac{1}{2.2}} & . & . & . & \sqrt{\frac{1}{2.n}}\\ . & . & . & & & .\\ . & . & & . & & .\\ . & . & & & . & .\\ \sqrt{\frac{1}{n.1}} & \sqrt{\frac{1}{n.2}} & . & . & . & \sqrt{\frac{1}{n.n}}\end{bmatrix}.$$

If Euclidean norm of the matrix A is computed, then we obtain

$$\left\|A^{\circ 1/2}\right\|_F^2=\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{1}{ij}\right)=(\Psi(n+1)+\gamma)\ \Psi(n+1)+(\Psi(n+1)+\gamma)\ \gamma$$

If we rearrange above expression, then we get

$$\left\|A^{\circ 1/2}\right\|_F^2=\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{1}{ij}\right)=(\Psi\,(\,n+1)+\gamma)^2\,.$$

Therefore we write,

(5) $$\left\|A^{\circ 1/2}\right\|_F=\Psi\,(\,n+1)+\gamma.$$

from (1) can deduce the following inequatity

$$\frac{1}{\sqrt{n}}\,(\Psi\,(\,n+1)+\gamma)\leqslant\left\|\left|A^{\circ(1/2)}\right|\right\|_2\leqslant\Psi\,(\,n+1)+\gamma$$

iii) The matrix $A^{\circ(-1)}$ is

$$A^{\circ(-1)}=\begin{bmatrix}1.1 & 1.2 & . & . & . & 1.n\\ 2.1 & 2.2 & . & . & . & 2.n\\ . & . & . & & & .\\ . & . & & . & & .\\ . & . & & & . & .\\ n.1 & n.2 & . & . & . & n.n\end{bmatrix}.$$

Thus we have

$$\begin{aligned}\left\|A^{\circ(-1)}\right\|_F^2 &= \left[(1.1)^2+(1.2)^2+...+(1.n)^2\right]+\left[(2.1)^2+(2.2)^2+...+(2.n)^2\right]\\ &\quad +...+\left[(n.1)^2+(n.2)^2+\ ...\ +(n.n)^2\right]\\ &= \sum_{k=1}^{n}(1.k)^2+\sum_{k=1}^{n}(2.k)^2+\ ...\ +\sum_{k=1}^{n}(n.k)^2\\ &= 1^2.\sum_{k=1}^{n}k^2+2^2.\sum_{k=1}^{n}k^2+\ ...\ +n^2.\sum_{k=1}^{n}k^2\\ &= \left[1^2+2^2+\ ...\ +n^2\right]\left(\sum_{k=1}^{n}k^2\right)=\left(\sum_{k=1}^{n}k^2\right)\left(\sum_{k=1}^{n}k^2\right)\\ &= \left(\sum_{k=1}^{n}k^2\right)^2\end{aligned}$$

from the definition of Euclidean norm and then we obtain

$$\left\|A^{\circ(-1)}\right\|_F=\sum_{k=1}^{n}k^2=\frac{n(n+1)(2n+1)}{6}.$$

Again from (1) we write

$$\frac{\sqrt{n}(n+1)(2n+1)}{6}\leqslant\left|\left|\left|A^{\circ(-1)}\right|\right|\right|_2\leqslant\frac{n(n+1)(2n+1)}{6}.$$

■

Theorem 3 *Let* $B=[\sqrt{ij}/(i+j)]_{i,j=1}^{n}$. *Then*

$$\begin{aligned}&\Big[\ (\Psi(2n+1)-\Psi(n+1))+n\left(\Psi(1,\ 2n+1)-\Psi(1,\ n+1)\right)\\ &+\frac{(n+1)(n+2)}{12n}-\Psi(1,2+n)\left(\frac{(2n+1)(n+1)}{6}\right)\\ &-\Psi(2+n)\left(\frac{3n^2+3n+1}{6n}\right)-\frac{\gamma}{6n}+\frac{f(n)}{n}\Big]^{1/2}\leq|||B|||_2\end{aligned}$$

and

$$\begin{aligned}|||B|||_2\leqslant\ &[n\left(\Psi(2n+1)-\Psi(n+1)\right)+n^2\left(\Psi(1,\ 2n+1)-\Psi(1,\ n+1)\right)+\\ &\frac{(n+1)(n+2)}{12}-n\Psi(1,\ 2+n)\left(\frac{(2n+1)(n+1)}{6}\right)-\\ &\Psi(2+n)\left(\frac{3n^2+3n+1}{6}\right)-\frac{\gamma}{6}+\ f(n)]^{1/2}\end{aligned}$$

hold where $f(n)\ =\ (\ 11031396).(\ 99961061\ )^n.10^{-8n-7}.n^{2208663.10^{-6}}$.

Proof. Since

$$B = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{3} & \frac{\sqrt{3}}{4} & . & . & . & \frac{\sqrt{n}}{n+1} \\ \frac{\sqrt{2}}{3} & \frac{1}{2} & \frac{\sqrt{6}}{5} & & & & \frac{\sqrt{2n}}{n+2} \\ \frac{\sqrt{3}}{4} & \frac{\sqrt{6}}{5} & \frac{1}{2} & & & & \frac{\sqrt{3n}}{n+3} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ \frac{\sqrt{n}}{n+1} & \frac{\sqrt{2n}}{n+2} & \frac{\sqrt{3n}}{n+3} & . & . & . & \frac{1}{2} \end{bmatrix},$$

we obtain

$$\begin{aligned} \|B\|_F^2 &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{ij}{(i+j)^2}\right) = n\,\Psi(2n+1) - \Psi(1,2n+1)n^2 + \\ & n^2\,(\Psi(1,2n+1)-\Psi(1,n+1)) - n\,\Psi(n+1) - \Psi(1,n+1)n^2 \\ & -\tfrac{1}{3}\Psi(1,2+n)\,(n+1)^3 - \Psi(2+n)\left(\tfrac{3n^2+3n+1}{6}\right) \\ & +\tfrac{1}{2}\left(\Psi(1,2+n)-\tfrac{1}{3}\right)(n+1)^2 + \left(-\tfrac{2}{3}-\tfrac{1}{6}\Psi(1,2+n)\right)(n+1) \\ & -\tfrac{1}{6}\Psi(2+n) - \tfrac{3}{4} - \tfrac{\gamma}{6} - \tfrac{1}{2}\Psi(2+n)(n+1)^2 + \tfrac{(n+1)^2}{4} - \tfrac{3n}{4} \\ & +\left(\tfrac{3}{2}+\tfrac{1}{2}\Psi(2+n)\right)(n+1) + \left(\sum_{i=1}^{n}(i^2\Psi(1,i+n)+i\Psi(i+n))\right) \end{aligned}$$

After rearranging this expression, we have

$$\begin{aligned} \|B\|_F^2 &= n\,(\Psi(2n+1)-\Psi(n+1)) + n^2\,(\Psi(1,2n+1)-\Psi(1,n+1)) \\ & -n\,\Psi(1,2+n)\left(\tfrac{(2n+1)(n+1)}{6}\right) - \Psi(2+n)\left(\tfrac{3n^2+3n+1}{6}\right) \\ & +\tfrac{(n+1)(n+2)}{12} - \tfrac{\gamma}{6} + \left(\sum_{i=1}^{n}(i^2\,\Psi(1,\ i+n) + i\,\Psi(i+n))\right). \end{aligned} \tag{6}$$

Let's try to find a function for the terms inside $\sum$ in (6). If we evaluate these terms, then we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}\left(i^2\Psi(1,\, i+n)+i\Psi(i+n)\right)\\
= \quad & \frac{1}{n}\{-\frac{(n+1)^3}{3}\Psi(1{,}2+n)+\frac{(n+1)^2}{2}\left(\Psi(1,\,2+n)-\frac{1}{3}\right)\\
& -n^2\Psi(1,\,n+1)+n^2\Psi(1,\,2n+1)-n\Psi(n+1)+n\Psi(2n+1)+\\
& (n+1)\left[-\frac{2}{3}-\frac{1}{6}\Psi(1,\,2+n)\right]-\frac{1}{6}\Psi(2+n)-\frac{3}{4}-\frac{1}{6}-\frac{(n+1)^2}{2}\Psi(2+n)\\
& +\frac{(n+1)^2}{4}-\frac{3}{4}n+(n+1)\left[\frac{3}{2}+\frac{1}{2}\Psi(2+n)\right]+1.1031396\ n^{\frac{11}{5}}\}\\
= \quad & \frac{1}{n}\{[\frac{(n+1)^2}{2}-\frac{(n+1)^3}{3}-\frac{n+1}{6}]\Psi(1,\,2+n)-n^2\Psi(1,\,n+1)+\\
& n^2\Psi(1,\,2n+1)-\left[\frac{1}{6}+\frac{(n+1)^2}{2}-\frac{(n+1)}{2}\right]\Psi(2+n)-\\
& n\Psi(n+1)+n\Psi(2n+1)-\frac{(n+1)^2}{6}-\frac{2}{3}(n+1)+\frac{3}{2}(n+1)\\
& -\frac{3}{4}-\frac{1}{6}\gamma+\frac{(n+1)^2}{4}-\frac{3}{4}n+1.1031396n^{\frac{11}{5}}\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{n}\{[\frac{(n+1)^2}{2}-\frac{(n+1)^3}{3}-\frac{n+1}{6}]\Psi(1,\,2+n)+n^2(\Psi(1,\,2n+1)-\Psi(1,\,n+1))\\
& -\left(\frac{(n+1)^2}{2}-\frac{n+1}{2}+\frac{1}{6}\right)\Psi(2+n)+n(\Psi(2n+1)-\Psi(n+1))\\
& +\frac{(n+1)^2}{12}+\frac{5}{6}(n+1)-\frac{3}{4}n-\frac{1}{6}\gamma-\frac{3}{4}+1.1031396n^{\frac{11}{5}}\}\\
= \quad & \frac{1}{n}\{-\frac{3n^2+3n+1}{6}\Psi(1,\,n+2)+n(\Psi(2n+1)-\Psi(n+1))+\frac{n^2+3n+2}{12}\\
& -\frac{1}{6}\gamma+1.1031396n^{\frac{11}{5}}\}
\end{aligned}
$$

and if this expression is rearranged, then we have

$$
\begin{aligned}
(7)\qquad \sum_{i=1}^{n}(i^2\Psi\,(1,i+n)+i\,\Psi\,(i+n)) \quad = \quad & -\frac{2n^3+3n^2+n}{6}\Psi(1{,}n+2)\\
& +n(\Psi(2n+1)-\Psi(n+1))\\
& +\left(\frac{1}{6n}-\frac{3n+3}{6}\right)\Psi(n+2)+\\
& \Psi(2n+1)-\Psi(n+1)+\frac{n+3}{12}\\
& +\frac{1}{6n}-\frac{1}{6n}\gamma+1.1031396n^{\frac{6}{5}}
\end{aligned}
$$

According to Table 1 for the different values of n in (7), the best proper function is

$$\begin{aligned} f(n) &= \sum_{i=1}^{n} \left(i^2\, \Psi(1,\ i+n) + i\, \Psi(\ i+n) \right) \\ &= (11031396).(99961061)^n.10^{-8n-7}.n^{2208663.10^{-6}} \end{aligned}$$

n = 1	1.067718	**n = 15**	432.27159
n = 2	4.96524	**n = 20**	816.459414
n = 3	12.18772	**n = 25**	1335.697984
n = 4	23.05798	**n = 30**	1995.485319
n = 5	37.82302	**n = 50**	6120.792804
n = 6	56.68172	**n = 60**	9118.075789
n = 7	79.80022	**n = 70**	12764.02713
n = 8	107.32113	**n = 80**	17074.00347
n = 9	139.36928	**n = 90**	22061.30790
n = 10	176.05570	**n = 100**	27737.67637

Table 1.

Hence we have

$$\begin{aligned} \|B\|_F^2 &= n\,(\Psi(2n+1) - \Psi(n+1)) + n^2\,(\Psi(1,2n+1) - \Psi(1,n+1)) \\ &\quad -n\,\Psi(1,2+n)\left(\tfrac{(2n+1)(n+1)}{6}\right) - \Psi(2+n)\left(\tfrac{3n^2+3n+1}{6}\right) \\ &\quad +\tfrac{(n+1)(n+2)}{12} - \tfrac{\gamma}{6} + f(n) \end{aligned} \tag{8}$$

Using (8) and (1), we have following inequalities:

$$\Big[(\Psi(2n+1) - \Psi(n+1)) + n\,(\Psi(1,2n+1) - \Psi(1,n+1)) + \tfrac{(n+1)(n+2)}{12n} - \Psi(1,2+n)\left(\tfrac{(2n+1)(n+1)}{6}\right) - \Psi(2+n)\left(\tfrac{3n^2+3n+1}{6n}\right) - \tfrac{\gamma}{6n} + \tfrac{f(n)}{n}\Big]^{1/2} \leqslant \||B\||_2$$

$$\||B\||_2 \leqslant \Big[n\,(\Psi(2n+1) - \Psi(n+1)) + n^2\,(\Psi(1,2n+1) - \Psi(1,n+1)) + \tfrac{(n+1)(n+2)}{12} - n\,\Psi(1,2+n)\left(\tfrac{(2n+1)(n+1)}{6}\right) - \Psi(2+n)\left(\tfrac{3n^2+3n+1}{6}\right) - \tfrac{\gamma}{6} + f(n)\Big]^{1/2}.$$

■

Corollary 1. For the matrix B given in Theorem 3, $B = B^{\circ(-1)}$.

3 Comparison of Bounds

In this section, we will be compared upper bounds obtained in Section 2 and upper bounds that will be obtained using inequalities in (5), and it will be determined which upper bounds are the best than the others.

Result 1. Let A be a matrix given in Theorem 2. If we use inequality (5) in Theorem 1, the following upper bounds will be satisfied

i) $|||A|||_2 \leqslant \frac{\pi^2}{6} - \Psi(1, n+1)$,

ii) $\left|\left|\left|A^{\circ(1/2)}\right|\right|\right|_2 \leqslant \Psi(n+1) + \gamma$,

iii) $\left|\left|\left|A^{\circ(-1)}\right|\right|\right|_2 \leqslant \frac{n\,(n+1)\,(2n+1)}{6}$

Proof. i) Let $A = B \circ C$ where

$$B = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \cdot & \cdot & \cdot & \frac{1}{n} \\ \frac{1}{1} & \frac{1}{2} & \cdot & \cdot & \cdot & \frac{1}{n} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \frac{1}{1} & \frac{1}{2} & \cdot & \cdot & \cdot & \frac{1}{n} \end{bmatrix} \text{ and } C = \begin{bmatrix} \frac{1}{1} & \frac{1}{1} & \cdot & \cdot & \cdot & \frac{1}{1} \\ \frac{1}{2} & \frac{1}{2} & \cdot & \cdot & \cdot & \frac{1}{2} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \frac{1}{n} & \frac{1}{n} & \cdot & \cdot & \cdot & \frac{1}{n} \end{bmatrix}.$$

Hence, since

$$r_1(B) = \sqrt{\left(\frac{1}{1}\right)^2 + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{n}\right)^2} = \sqrt{\sum_{k=1}^{n} \frac{1}{k^2}} = \sqrt{\frac{\pi^2}{6} - \Psi(1, n+1)}$$

and

$$c_1(C) = \sqrt{\left(\frac{1}{1}\right)^2 + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{n}\right)^2} = \sqrt{\sum_{k=1}^{n} \frac{1}{k^2}} = \sqrt{\frac{\pi^2}{6} - \Psi(1, n+1)},$$

we have

$$|||A|||_2 \leqslant r_1(B)\, c_1(C) = \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \Psi(1, n+1). \tag{9}$$

It is seen that the upper bound in (9) is the same in Theorem 2 (i). Consequently, the upper bound obtained in Theorem 2 (i) is the best upper bound.

ii) Let $A^{\circ(1/2)} = M \circ P$ where

$$M = \begin{bmatrix} \frac{1}{1} & \frac{1}{\sqrt{2}} & . & . & . & \frac{1}{\sqrt{n}} \\ \frac{1}{1} & \frac{1}{\sqrt{2}} & . & . & . & \frac{1}{\sqrt{n}} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \frac{1}{1} & \frac{1}{\sqrt{2}} & . & . & . & \frac{1}{\sqrt{n}} \end{bmatrix} \text{ and } P = \begin{bmatrix} \frac{1}{1} & \frac{1}{1} & . & . & . & \frac{1}{1} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & . & . & . & \frac{1}{\sqrt{2}} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & . & . & . & \frac{1}{\sqrt{n}} \end{bmatrix}.$$

Hence since

$$\begin{aligned} r_1(M) &= \sqrt{\left(\tfrac{1}{1}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 + \ldots + \left(\tfrac{1}{\sqrt{n}}\right)^2} = \sqrt{\tfrac{1}{1} + \tfrac{1}{2} + \ldots + \tfrac{1}{n}} \\ &= \sqrt{\sum_{k=1}^{n} \tfrac{1}{k}} = \sqrt{\Psi\,(\,n+1\,) \,+\, \gamma} \end{aligned}$$

and

$$\begin{aligned} c_1(P) &= \sqrt{\left(\tfrac{1}{1}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 + \ldots + \left(\tfrac{1}{\sqrt{n}}\right)^2} = \sqrt{\tfrac{1}{1} + \tfrac{1}{2} + \ldots + \tfrac{1}{n}} \\ &= \sqrt{\sum_{k=1}^{n} \tfrac{1}{k}} = \sqrt{\Psi\,(\,n+1\,) \,+\, \gamma}, \end{aligned}$$

we have

$$\text{(10)} \quad \begin{aligned} \left|\left|\left| A^{\circ(1/2)} \right|\right|\right|_2 &\leqslant r_1\,(M)\, c_1\,(P) = \sqrt{\Psi(n+1) + \gamma}\,\sqrt{\Psi\,(n+1) + \gamma} \\ &= \Psi\,(n+1) + \gamma \end{aligned}$$

It is seen that the upper bound in (10) is the same in Theorem 2 (ii). Consequently, the upper bound obtained in Theorem 2 (ii) is the best upper bound.

iii) Let $A^{\circ(-1)} = K \circ L$ where

$$K = \begin{bmatrix} 1 & 2 & . & . & . & n \\ 1 & 2 & . & . & . & n \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 1 & 2 & . & . & . & n \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 1 & . & . & . & 1 \\ 2 & 2 & . & . & . & 2 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ n & n & . & . & . & n \end{bmatrix}.$$

Since

$$r_1(K) = \sqrt{1^2 + 2^2 + \ldots + n^2} = \sqrt{\sum_{k=1}^{n} k^2} = \sqrt{\frac{n\,(n+1)\,(2n+1)}{6}}$$

and

$$c_1(L) \;=\; \sqrt{1^2 + 2^2 + \ldots + n^2} = \sqrt{\sum_{k=1}^{n} k^2} \;=\; \sqrt{\frac{n\,(n+1)\,(2n+1)}{6}},$$

we obtain

$$\left|\left|\left| A^{\circ(-1)} \right|\right|\right|_2 \leqslant\; r_1(K)\, c_1(L) \;=\; \frac{n\,(n+1)\,(2n+1)}{6}. \tag{11}$$

It is seen that the upper bound in (11) is the same in Theorem 2 (iii). Consequently, the upper bound obtained in Theorem 2 (iii) is the best upper bound. ■

Result 2. Let $B = [\sqrt{ij}/(i+j)]_{i,j=1}^{n}$. Then

$$|||B|||_2 \leqslant\; n\sqrt{\frac{n+1}{2}\left(\frac{\pi^2}{6} - 1 - \Psi\,(\,1\,,\,n+2\,)\right)}.$$

Proof. Let $B = C \circ D$ where

$$C \;=\; \begin{bmatrix} \frac{\sqrt{1}}{1} & \frac{\sqrt{2}}{1} & \cdot & \cdot & \cdot & \frac{\sqrt{n}}{1} \\ \frac{\sqrt{2}}{1} & \frac{\sqrt{4}}{1} & \cdot & \cdot & \cdot & \frac{\sqrt{2n}}{1} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \frac{\sqrt{n}}{1} & \frac{\sqrt{2n}}{1} & \cdot & \cdot & \cdot & \frac{\sqrt{n^2}}{1} \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \cdot & \cdot & \cdot & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \cdot & \cdot & \cdot & \frac{1}{n+2} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdot & \cdot & \cdot & \frac{1}{n+n} \end{bmatrix}.$$

Since

$$\begin{aligned} r_n(C) \;&=\; \sqrt{n + 2n + \ldots + n^2} \;=\; \sqrt{n(1+2+\ldots+n)} \;= \\ &=\; \sqrt{n\,\frac{n(n+1)}{2}} = n\sqrt{\frac{n+1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) \;&=\; \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \ldots + \left(\frac{1}{n+1}\right)^2} \;= \\ &=\; \sqrt{\sum_{k=1}^{n} \frac{1}{(k+1)^2}} = \sqrt{\frac{\pi^2}{6} - 1 - \Psi(\,1\,,\,n+2\,)}, \end{aligned}$$

we have

$$|||B|||_2 \leqslant\; r_n(C)\, c_1(D) \;=\; n\sqrt{\frac{n+1}{2}\left(\frac{\pi^2}{6} - 1 - \Psi\,(\,1\,,\,n+2\,)\right)}. \tag{12}$$

It is seen that the upper bound in Theorem 3 is better than (12)

■

4. Numerical Results

Table 2.

n	$\frac{\pi^2}{6\sqrt{n}}$	$\lvert\lvert\lvert A\rvert\rvert\rvert_2$	$\frac{\pi^2}{6}$
5	0.7356368786	1.463611111	1.644934068
10	0.5201738254	1.549767731	1.644934068
15	0.4247201498	1.580440283	1.644934068
20	0.3678184394	1.596163244	1.644934068
30	0.3003224981	1.612150118	1.644934068
40	0.2600869127	1.620243963	1.644934068
50	0.2326288067	1.625132734	1.644934068
100	0.1644934068	1.634983900	1.644934068
150	0.1343083042	1.638289573	1.644934068

Table 3.

n	$\frac{1}{\sqrt{n}}(\Psi(n+1)+\gamma)$	$\lvert\lvert\lvert A^{\circ(1/2)}\rvert\rvert\rvert_2$	$\Psi(n+1)+\gamma$
5	1.1021137710	2.283333333	2.283333333
10	0.9262210876	2.928968254	2.928968254
15	0.8567630419	3.318228993	3.318228993
20	0.8044790440	3.597739657	3.597739657
30	0.7293815229	3.994987131	3.994987131
40	0.6764970533	4.278543039	4.278543039
50	0.6362837207	4.499205338	4.499205338
100	0.5187377518	5.187377518	5.187377518
150	0.4565179833	5.591180589	5.591180589

Table 4.

n	$\frac{1}{\sqrt{n}}(\Psi(n+1)+\gamma)$	$\lvert\lvert\lvert A^{\circ(1/2)}\rvert\rvert\rvert_2$	$\Psi(n+1)+\gamma$
5	1.1021137710	2.283333333	2.283333333
10	0.9262210876	2.928968254	2.928968254
15	0.8567630419	3.318228993	3.318228993
20	0.8044790440	3.597739657	3.318228993
30	0.7293815229	3.994987131	3.994987131
40	0.6764970533	4.278543039	4.278543039
50	0.6362837207	4.499205338	4.499205338
100	0.5187377518	5.187377518	5.187377518
150	0.4565179833	5.591180589	5.591180589

Table 5.

N	$\frac{K(n)}{\sqrt{n}}$	$\|\|B\|\|_2$	$K(n)$
5	1.108080674	2.329355055	2.477743720
10	1.496873548	4.541330816	4.733529779
15	1.784423184	6.736973533	6.911041296
20	2.026826140	8.927158040	9.0642422197
30	2.441758874	13.30120029	13.37406408
40	2.802603736	17.67175325	17.72522214
50	3.128091487	22.04080444	22.11894684
100	4.143819216	43.87888779	44.13819216
150	5.284079187	64.71362680	64.71648868

where

$$K(n) = [n\left(\Psi(2n+1) - \Psi(n+1)\right) + n^2\left(\Psi(1,\,2n+1) - \Psi(1,\,n+1)\right) + \frac{(n+1)(n+2)}{12} - n\Psi(1,\;2+n)\left(\frac{(2n+1)(n+1)}{6}\right) - \Psi(2+n)\left(\frac{3n^2+3n+1}{6}\right) - \frac{\gamma}{6} + f(n)]^{1/2}$$

Table 6.

N	$\|\|B\|\|_2$	$n\sqrt{\frac{n+1}{2}\left(\frac{\pi^2}{6} - 1 - \Psi(1,\,n+2)\right)}$
5	2.329355055	6.070763269
10	4.541330816	17.51906695
15	6.736973533	32.43183251
20	8.927158040	50.13391502
30	13.30120029	92.48789255
40	17.67175325	142.7007855
50	22.04080444	199.6915662
100	43.87888779	566.3182628
150	64.71362680	1041.330359

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