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Optimum spectral parameter and convergency for stationary iterative methods in the case of three-diagonal SLAE*

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Summary. The modified stationary iterative methods of the solution of system of the linear algebraic equations (SLAE) are considered. For SLAE with a three-diagonal matrix with constant factors it is shown, that eigenvalues of modified matrices or the operator, participating in series of simple iteration, are expressed through roots of Chebyshev polynomials of the second kind. On this basis strict expressions through factors of an initial matrix for optimum parameter of convergence and spectral radius are found. So for Successive Overrelaxation method strict expression for the optimum parameter of convergence ω_0 laying on an interval $(0, 2)$ is found. It is shown, that convergence of the optimum modified series essentially improves.

Key words: stationary iterative methods, spectral radius, matrix equations

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The numerical iterative methods of the solution of systems of linear algebraic equations (SLAE)

$$Ax = b \tag{1}$$

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are esteemed. Here $A = (a_{ij})$, $i, j = 1, 2, \ldots, n$ is an n-dimensional regular matrix (it is specify by notes to formula (8)), $b \in C^n$ is a given vector, x is a vector in the question.

Many iterative methods can be shown to process of simple iteration. Thus the input equation by that or different way should be shown to an equation

$$x = Bx + z. \tag{2}$$

Here x- unknown vector, z- given vector on the right of the line of equation, B - given matrix of factors. For example, if SLAE (1) is set, directly receiving

$$B = I - A, \tag{3}$$

where I- unit matrix, we come to (2). Let's remark, that the transition from (1) to (2) can be executed not by an alone way, that results in different modifications of a method of simple iteration - Richardson method, Jacobi method, Gauss-Seidel method, etc. [1]

The process of simple iteration is as follows:

$$x^{(m+1)} = Bx^{(m)} + b, m = 0, 1, \ldots. \tag{4}$$

Generally, the initial guess to the solution is $x^{(0)} = b$.

It is demonstrated [2], that an indispensable and sufficient condition of convergence of process of simple iteration (4) is

$$\rho(B) < 1, \tag{5}$$

where $\rho = \rho(B) = \max_i |\beta_i|$ - spectral radius of a matrix B, β_i - eigenvalues of matrix B. Thus the iterations converge not worse than geometrical progression with a denominator $q = \rho(B)$.

Thus, on acceleration of convergence of a method of simple iteration, the equation (1) we shall write down equivalently

$$x = B_k x + \frac{b}{1+k}, \tag{6}$$

where the matrix B_k is determined by the formula

$$B_k = \frac{1}{1+k}(B + kI). \tag{7}$$

Here k- while any, $k \neq -1$, complex parameter which choice we shall try to satisfy a condition $\rho(B_k) < \rho(B)$ or $\rho(B_k) < 1$ in case (5) it is not executed.

The approach (6), (7) has been successfully applied for acceleration of convergence (4) in case, when B- the linear continuous operator in Banah space [3].

Thus we yet do not consider an opportunity of occurrence so-called ε- spectrum at a matrix [4]. In other words, it is supposed, that the machine constant ε_{comp} is chosen enough small that it was possible to neglect an opportunity of occurrence of a pseudo-spectrum.

Let's consider SLAE (1) with a three-diagonal matrix A and B of a kind

$$(8) \qquad A = \begin{pmatrix} a_0 & a_1 & & & \\ a_{-1} & a_0 & a_1 & O & \\ & \ddots & \ddots & \ddots & \\ & O & a_{-1} & a_0 & a_1 \\ & & & a_{-1} & a_0 \end{pmatrix}, B = \begin{pmatrix} d & b & & & \\ a & d & b & O & \\ & \ddots & \ddots & \ddots & \\ & O & a & d & b \\ & & & a & d \end{pmatrix}.$$

If in (4) B- the operator, instead of a matrix, as in Seidel method, the matrix, its generating has the same three-diagonal structure (8). Everywhere further it is supposed, as product ab is real and in items 1, 2 a_0 is real too.

The matrix of such kind frequently arises at the solution of the ordinary differential equations, and also is a component of matrixes at the solution of initial-boundary value problems for the differential equations.

Let's find strict conditions of convergence and optimum parameters for acceleration of convergence for stationary iterative methods at the solution of SLAE with such matrix.

1. Method of simple iteration (Richardson method)

If in a method of simple iteration it is applied (3) it refers to as a Richardson method [1]. In this case $d = 1 - a_0$, $a = -a_{-1}$, $b = -a_1$.

Let's find a spectrum of a square n-dimensional matrix B. The characteristic equation $Bx = \lambda x$ results in the equation $d_n = 0$, where a determinant d_n of a kind

$$(1.1) \qquad d_n = \det(B - \lambda I).$$

Displaying a determinant on the first line, it is easy to receive the recurrent formula

$$(1.2) \qquad d_n = (d - \lambda)d_{n-1} - abd_{n-2}, n = 3, 4, \ldots.$$

Initial values follow from (1.1): $d_1 = d - \lambda$, $d_2 = (d - \lambda)^2 - ab$. Further, with the help (1.2), we receive $d_3 = (d - \lambda)((d - \lambda)^2 - 2ab)$, $d_4 = (d - \lambda)^4 - 3ab(d - \lambda)^2 + a^2 b^2$ and so on.

Let's enter replacement of a variable

$$(d - \lambda)^2 = \mu ab. \tag{1.3}$$

Then determinants correspond as

$$d_1 = d - \lambda,$$

$$d_2 = (\mu - 1)ab,$$

$$d_3 = (d - \lambda)(\mu - 2)ab,$$

$$d_4 = (d - \lambda)^2(\mu - 2)ab - (\mu - 1)a^2b^2 = (\mu^2 - 3\mu + 1)a^2b^2,$$

etc.

It is easy to notice, that each odd determinant, due to (1.2) has a multiplier $(d-\lambda)$, and everyone even due to replacement of a variable (1.3) raises a degree μ on unit. I.e. $(m = 1, 2, \ldots)$,

$$d_{2m} = (ab)^m P_m(\mu);\ d_{2m+1} = (d - \lambda)(ab)^m Q_m(\mu), \tag{1.4}$$

where $P_m(\mu)$ and $Q_m(\mu)$ - polynomials of a variable μ degree m. For them, taking into account (1.2), we receive recurrent formulas $(m = 1, 2, \ldots)$,

$$P_m(\mu) = \mu Q_{m-1}(\mu) - P_{m-1}(\mu);\ Q_m(\mu) = P_m(\mu) - Q_{m-1}(\mu). \tag{1.5}$$

Whence it is easy to receive recurrent formula only for polynomials

$$Q_m(\mu) = (\mu - 2)Q_{m-1}(\mu) - Q_{m-2}(\mu),\ m = 3, 4, \ldots. \tag{1.6}$$

and

$$Q_1(\mu) = \mu - 2, Q_2(\mu) = \mu^2 - 4\mu + 3, \ldots.$$

Let's make replacement of a variable

$$\mu = 2(x + 1). \tag{1.7}$$

Then polynomials (1.6) pass in the following

$$t_m(x) = 2xt_{m-1}(x) - t_{m-2}(x),\ m = 3, 4, \ldots \tag{1.8}$$

and $t_1(x) = 2x$; $t_2(x) = 4x^2 - 1, \ldots$.

These are well-known orthogonal Chebyshev polynomials of the second kind with roots on interval $x \in (-1, 1)$ [2].

$$t_m(x) = \sin((m + 1)\arccos(x))/\sqrt{1 - x^2}. \tag{1.9}$$

Hence, (1.6) - the same Chebyshev polynomials of the second kind, but with roots on a piece $\mu \in (0, 4)$.

Thus, for eigenvalues of a matrix B the following theorem is fair

Theorem 1. *Eigenvalues of $n-$ dimensional matrix B (3), (8) are:*

1. $\lambda_\nu^{(1,2)} = d \pm \sqrt{\mu_\nu ab}$,

$$\mu_\nu = 2(x_\nu + 1), x_\nu = \cos\left(\frac{2\pi\nu}{n+1}\right), \nu = 1, 2, \ldots, [n/2]. \tag{1.10}$$

2. In case of odd n there exists an additional root $\lambda_0 = d$.
Here μ_ν, x_ν- roots of Chebyshev polynomial of second kind (1.8), (1.9) in case of odd value of $n = 2m+1, m = 1, 2, \ldots$ and roots of equation $t_{m-1}(x) + t_m(x) = 0$ in case of even value $n = 2m$, $m = 1, 2, \ldots$; $[x]$ is an integer part function of x.

The proof. By virtue the first formula of (1.4), second one of (1.5) and (1.6 - 1.8) roots of even determinants $n = 2m$, $m = 1, 2, \ldots$ are roots of the equation $t_{m-1}(x) + t_m(x) = 0$. Taking into account (1.9), we receive the equation $\sin((m + 0.5) \arccos x) = 0$ for all roots, except for $x = 1$ which in view of feature in a denominator (1.9) results in finite value $m + 0.5$ and which should be rejected therefore. We received for roots in this case $(m+0.5) \arccos x = \pi\nu$, $\nu = 1, 2, \ldots, m$, $m = n/2$ and from here follows (1.10).

For odd determinants $n = 2m + 1$, $m = 1, 2, \ldots$ valid (1.4), (1.6), (1.9) it is received $d_{2m+1} = 0$, and therefore $\lambda_0 = d$ and $t_m(x) = 0$, i.e. $(m + 1) \arccos x = \pi\nu, \nu = 1, 2, \ldots, m$, $m = [n/2]$. From values ν are excluded $\nu = 0$, $\nu = m+1$, since disclosing of uncertainty in (1.9) at $x = \pm 1$ results in finite, nonzero value. In result for roots in this case it is received (1.10). Further, with the aid of (1.3) the theorem is proved.

Consequence 1. Spectral radius of a matrix (3), (8) is

$$\rho(\beta) = \max_\nu |\lambda_\nu| = \begin{cases} |d| + \sqrt{\mu_{max} ab}, & \text{for } ab > 0; \\ \sqrt{|d|^2 - \mu_{max} ab}, & \text{for } ab < 0. \end{cases} \tag{1.11}$$

Really, in figure 1 the typical behavior of two branches of function $f(x) = |d \pm \sqrt{\mu x}|$ is shown. From here follows, by virtue of monotonous increase of the top branches of function, since $x = 0$, that $\max_\nu |\lambda_\nu|$ comes for the maximal root μ_ν on an interval (0,4) at $\nu = 1$. I.e. for $\mu_1 = \mu_{max} = 2(x_{max} + 1)$ and

$$x_{max} = \cos\left(\frac{2\pi}{n+1}\right) \tag{1.12}$$

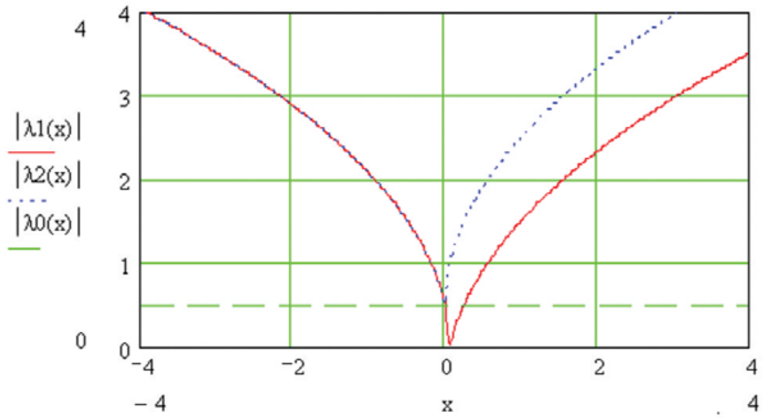

Fig. 1.

Here $x_{max} = x_1$ is a maximal root of Chebyshev polynomial $t_m(x)$ of second kind (1.8), (1.9) in case of odd value of $n = 2m+1, m = 1, 2, \ldots$ and it is the maximal root of equation $t_{m-1}(x) + t_m(x) = 0$ in case of even value $n = 2m$, $m = 1, 2, \ldots$.

Further, the analysis of behavior $|\lambda_{max}|$ in dependence from ab results to (1.11). In figure 1 value $|\lambda_0| = |d|$ which is not considered for (1.12) because of it is always less than value of the greater branch is shown also.

Consequence 2. Convergence of a method of simple iteration for a matrix (3),(8) takes place only for $|d| < 1$ and thus for very narrow circle of the values ab set (1.11) and $\rho < 1$. For the big matrices $n \to \infty$ in (1.11) it is necessary to substitute $\mu_{max} \to 4$.

2. Optimal method of simple iteration (Optimal Richardson method)

For matrix (7) similarly to how it has been made in item 1, we receive

Theorem 2. *Eigenvalues of $n-$ dimensional matrix (3), (8), (7) are* $(m = 1, 2, \ldots)$*:*

$$\lambda_\nu^{(1,2)} = \frac{1}{k+1}(d + k \pm \sqrt{\mu_\nu ab}); \tag{2.1}$$

2. In case of odd n there exists an additional root

$$\lambda_0 = \frac{d+k}{1+k}.$$

Here $\mu_\nu = 2(x_\nu + 1)$ and x_ν sets by formula (1.10).

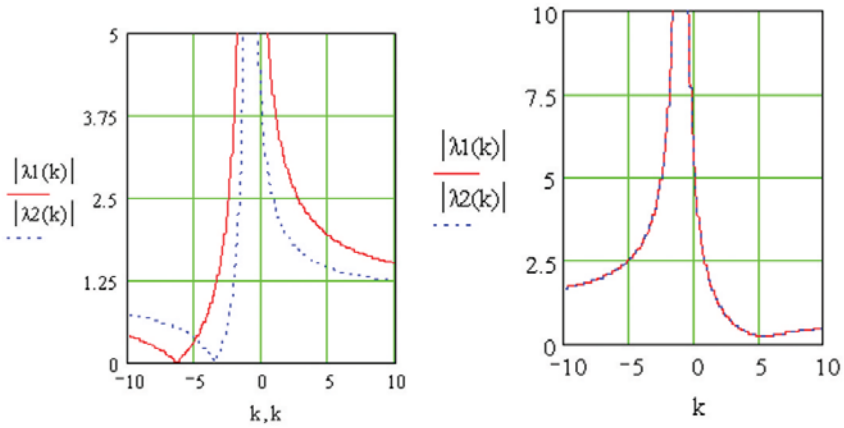

Fig. 2.

Eigenvalue λ_0 by search of optimal parameter for convergence (1.4) with $B = B_k$ can be rejected, because $|\lambda_0| \leq |\lambda_\nu|$.

Theorem 3. *Optimal parameter for convergence (1.4) with a matrix (3),(8),(7) and spectral radius of a matrix at this optimal parameter are:*

$$\begin{array}{lll} & \textit{1. For } ab > 0 \; k_0 = -d, & \rho(B_{k_0}) = \sqrt{\mu_{max} ab}/|d-1|; \\ (2.2) & \textit{2. for } \; ab < 0 \; k_0 = -d + \frac{\mu_{max} ab}{1-d}, & \rho(B_{k_0}) = \frac{1}{\sqrt{1 - \frac{(1-d)^2}{\mu_{max} ab}}}. \end{array}$$

Really, as well as at the proof of the theorem 1, $\max_\nu |\lambda_\nu|$ comes for the maximal root μ_{max}. Further, in fig. 2 the typical behavior of two branches of function $|\lambda_{max}|$ from parameter k in (2.1) and definitions of spectral radius k_0 in cases $ab > 0$ and $ab < 0$ are submitted. In a case $ab > 0$ for a choice k_0 it is necessary to take a point of crossing of two branches of function that leads to item 1 in (2.2), and in a case $ab < 0$ both branches coincide and it is necessary to find a minimum with the help of a derivative that leads to item 2 in (2.2).

Consequence. The optimal parameter for convergence of the modified method of simple iteration in a case $ab > 0$ does not depend on factors a, b and is given by the simple formula $k_0 = -d = a_0 - 1$. Convergence takes place at a ratio between factors $\sqrt{\mu_m ab} < |1 - d|$, i.e. in terms of matrix A:

$$\mu_m a_{-1} a_1 < a_0^2. \tag{2.3}$$

In a case $ab < 0$ convergence at optimal parameter takes place for any a, b.

3. The Jacobi, Gauss-Seidel and Optimal Successive Overrelaxation method

We shall present $A = D+L+U$, where D, L, U represent the diagonal, the strictly lower-triangular and the strictly upper-triangular parts of a matrix A (1), respectively.

The Gauss-Seidel method [1] assumes recording the initial equation $Ax = b$ as

$$x = Bx + z, \tag{3.1}$$

where $B = -D^{-1}(L+U)$ - a matrix with a zero main diagonal $d = 0$ with factors $b_{ij} = -\frac{a_{ij}}{a_{ii}}$, $i, j = 1, 2, \ldots, n$; $i \neq j$. Components of a vector $z_i = \frac{y_i}{a_{ii}}$. Further process of iterations is as follows

$$x^{(m+1)} = \hat{B}_S x^{(m)} + z, m = 0, 1, \ldots . \tag{3.2}$$

Here $\hat{B}_S$ - the Gauss-Seidel (GS) operator, which influence on a vector of the previous iteration $x^{(k)}$ is divided on two parts, in first of which components of a vector $x^{(k+1)}$ already found on the current iteration are used

$$\hat{B}_S x^{(m)} = -D^{-1}Lx^{(m+1)} - D^{-1}Ux^{(m)}. \tag{3.3}$$

If in (3.2) the matrix $B = -D^{-1}(L+U)$, instead of the operator (3.3) is used, such method is known as Jacobi method [1]. In case of three-diagonal SLAE conditions of convergence of usual and optimal Jacobi methods, and also spectral radii and optimal parameter it is possible to receive from formulas (1.11) and (2.2) at $d = 0$. We shall take into account, that in Jacobi method factors of matrix (8) are $a = -a_{-1}/a_0$, $b = -a_1/a_0$. The analysis of the above formulas shows that usual Richardson method conceded in the domain of convergence to usual Jacobi method whereas conditions of convergence and spectral radii for optimum Richardson and Jacobi methods coincide.

The result of influence of the operator $\hat{B}_S$ on a vector $x^{(m)}$, i.e. a vector $\hat{B}_S x^{(m)}$, can be received by multiplication of some matrix B_S on $x^{(m)}$. Then $B_S = -(D+L)^{-1}U$ and $z = -(D+L)^{-1}b$

$$\hat{B}_S x^{(m)} = B_S x^{(m)}. \tag{3.4}$$

However, processes occurring in the left and right parts of equality (3.4) are essentially various. To notice, that the matrix B (3.1), which generate the operator $\hat{B}_S$, in case of three-diagonal SLAE has the main zero diagonal and two nonzero diagonals, but the matrix

B_S contains the lower triangular with unequal to zero the main diagonal and with equal to zero the first column matrix . Quantity of arithmetic operations of multiplication at the left and on the right in (5) are also essentially various. So, it is - $2(n-1)$ and $n(n+1)/2-1$ respectively.

Nevertheless, it is possible to show that the operator has the same eigenvalues, as a matrix in (3.4) (at least, for a three-diagonal case under consideration). However, if to set the task of acceleration of convergence (3.2) the optimum spectral parameter for a matrix in (3.4) and the optimum parameter for the operator (3.3) as will be shown further, various.

Let B be a matrix with any (generally speaking, distinct from zero) the main diagonal. Action of the operator $\hat{B}$ generated by this matrix, we shall determine by analogy with (3.3) with that distinction, that U - the upper- triangular matrix including the main diagonal. Thus, new coordinates of a vector are determined as

$$x_i = \sum_{j=1}^{n} b_{ij} x_j, \qquad i = 1, 2, \ldots, n. \tag{3.5}$$

And old coordinates in the right part (3.5) in process of growth i are replaced on new, found in the left part.

For acceleration of convergence (4) we shall use (6), (7), where B_k we should replace on $\hat{B}_k$ - modified GS operator corresponding to matrix (7)

$$\hat{B}_k = \frac{1}{1+k}(\hat{B} + kI). \tag{3.6}$$

In the matrix form the result of influence of the operator (3.6) can be received with the help of the following matrix

$$B_k = (L + (1+k)D)^{-1}(kD - U). \tag{3.7}$$

The set vector of the right part is transformed thus in

$$b_k = (L + (1+k)D)^{-1} b.$$

As well as in case of a usual method at $k = 0$ the matrix (3.7) has the same spectrum, as the operator (3.6).

An indispensable and sufficient condition

$$\rho(B) < 1, \ \rho(B) = \max_i |\beta_i|$$

of convergence of process of simple iteration (4) are fair as the matrix (3.7) with the same spectrum and resulting action is known, as well as at the operator (3.6) [2].

Let's consider a spectrum of modified GS operator (3.6) in three-diagonal case (8), and also conditions of convergence (4) at its participation. Rename for simplicity of recording

$$k/(1+k) \to k,\ a/(1+k) \to a,\ b/(1+k) \to b. \tag{3.8}$$

Lemma 1. *Determinants d_n of the characteristic equation $\hat{B}_k x = \lambda x$ with the operator (3.6) satisfy to the recurrent formula*

$$d_n = (k-\lambda)d_{n-1} - ab\lambda d_{n-2}.$$

Really, the determinant of the characteristic equation generally looks like

$$d_n = \begin{vmatrix} k-\lambda & b & 0 & \dots & 0 \\ ak & ab+k-\lambda & b & \dots & 0 \\ a^2k & ab+k & ab+k-\lambda & \dots & 0 \\ a^3k & a^2b+k & ab+k & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a^{n-2}k & a^{n-3}b+a^{n-4}k & a^{n-4}b+a^{n-5}k & \dots & b \\ a^{n-1}k & a^{n-2}b+a^{n-3}k & a^{n-3}b+a^{n-4}k & \dots & ab+k-\lambda \end{vmatrix}.$$

Displaying a determinant d_n on the last column, we receive

$$d_n = (ab+k-\lambda)d_{n-1} - ba\tilde{d}_{n-1}.$$

Here $\tilde{d}_{n-1}$ - a determinant which differs from d_{n-1} ones, that at it last element is equal $ab+k$ instead of $ab+k-\lambda$. Thus, $\tilde{d}_{n-1} = d_{n-1} + \lambda d_{n-2}$ and, substituting it in d_n, we receive the formulation of Lemma.

For example, at $n = 2$, the characteristic equation looks like

$$\begin{cases} kx_1 + bx_2 = \lambda x_1, \\ a(kx_1 + bx_2) + kx_2 = \lambda x_2. \end{cases}$$

A determinant $d_2 = (k-\lambda)^2 - ab\lambda$. Eigenvalues in this case, taking into account (3.8)

$$\lambda_{1,2}(k) = \frac{k}{k+1} + \frac{ab}{2(k+1)^2}(1 \pm \sqrt{1 + \frac{4k(k+1)}{ab}}). \tag{3.9}$$

The problem of acceleration of convergence will consist in minimization of spectral radius as functions of a variable k, i.e. it is necessary to find optimum parameter k_0 at which there comes a minimum

$$\rho_0(B) = \min_k \rho(B,k) = \min_k \max_{\nu=1,2} |\lambda_\nu(k)| < 1. \tag{3.10}$$

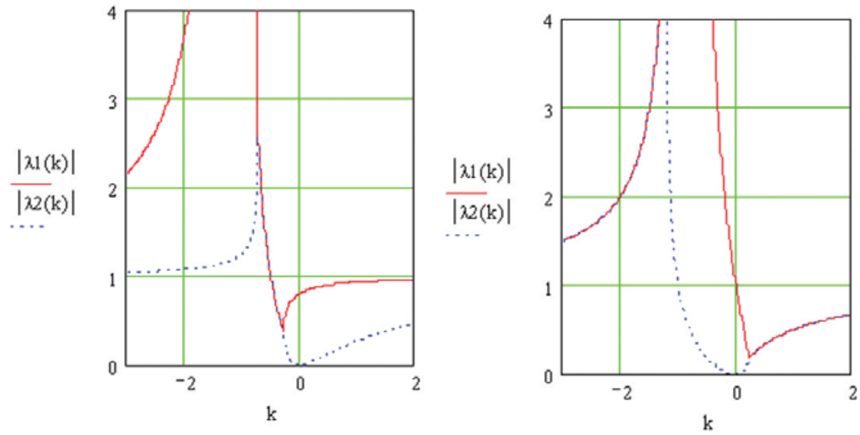

Fig. 3.

Research of function $f(k) = |\lambda_i(k)|$ shows, that the minimum (3.10) comes at real k, one of roots of a radicand in (3.9):

$$(3.11) \qquad k_0 = \frac{1}{2}(\sqrt{1-ab}-1),\ \rho_0(B) = |\lambda_i(k_0)| = \frac{|1-\sqrt{1-ab}|^2}{|ab|}.$$

Really, the behavior of both branches of function $|\lambda_i(k)|$ depending on k for a cases $ab > 0$ and $ab < 0$ is shown on fig. 3. It is possible to show, that if the radicand in (3.9) has no roots which correspond to k (it is a case $ab > 1$) branches $|\lambda_i(k)|$ lay on the different sides from a straight line $\lambda = 1$. If has one root (it is a case $ab = 1$) branches are crossed in one point on $\lambda = 1$. In these cases convergence is not present. If $ab < 1$, the radicand in (3.9) has two roots and $|\lambda_i(k_1)\lambda_i(k_2)| = 1$, and for the right root $k_2 = k_0$ (3.11) is carried out $|\lambda_i(k_0)| < 1$. In points $k = k_1$ and $k = k_2$ two branches of function merge in one and behave in dependence from ab as shown in fig. 3.

Thus, value $\rho_0 < 1$ for everything $ab < 1$, that is much wider, than for usual GS method, for which from (3.9) we have at $k = 0$ two eigenvalues $\lambda_0 = 0$, $\lambda_1 = ab$ and the spectral radius $\rho(B) = |ab|$. So, the relation $\rho_0(B)/\rho(B) = \rho_0(B)/|ab| < 1$ for $0 < ab < 1$ and it is $\ll 1$ for $ab < 0$.

For a case on the right part fig. 3; $a = -1$, $b = 1$ and for this case $k_0 = 0.207$, $|\lambda(k_0)| = 0.172$, that testify too fast convergence of the modified series. Thus $f(0) = 1$ and usual GS series does not converge.

For a case $n = 3$ it is similarly received, that a determinant is

$$d_3 = (k-\lambda)((k-\lambda)^2 - 2ab\lambda),$$

optimum parameter is $k_0 = (\sqrt{1-2ab} - 1)/2$ and radius of convergence is

$$\rho_0(B) = |\sqrt{1-2ab} - 1|^2/(2|ab|).$$

The inequality $\rho_0(B) < 1$ is value when $ab < 0.5$. The root $\lambda = k/(k+1)$ arising for odd determinants, is not taken into account, as at $k = k_0$ is carried out $|k_0/(k_0+1)| = \rho_0(B)$.

Using a recurrent formula (18), we receive expression for the following determinants

$$d_4 = (k-\lambda)^4 - 3ab\lambda((k-\lambda)^2 + (ab\lambda)^2,$$

$$d_5 = (k-\lambda)((k-\lambda)^4 - 4ab\lambda((k-\lambda)^2 + 3(ab\lambda)^2),$$

etc. We shall notice, that the determinants are the polynomials into which even degrees of $(k-\lambda)$ enter only, and in odd polynomials there is a general multiplier $(k-\lambda)$.

Let's lead replacement of a variable λ on μ as follows

$$(k-\lambda)^2 = \mu ab\lambda. \tag{3.12}$$

Note that $\lambda \neq 0$ if $k \neq 0$. In result for determinants it is received $d_{2m} = (ab\lambda)^m P_m(\mu)$ and $d_{2m+1} = (k-\lambda)(ab\lambda)^m Q_m(\mu)$, $m = 1, 2, \ldots$. For these polynomials $P_m(\mu)$ and $Q_m(\mu)$ in view of the Lemma 1 it is received recurrent formula (1.5), whence it is easy to receive recurrent formula for polynomials $Q_m(\mu)$ (1.6). Replacement of a variable (1.7) translates polynomials $Q_m(\mu)$ in polynomials (1.8), (1.9) - $t_m(x)$, that testifies to that, what is it the Chebyshev polynomials of second kind with roots on intervals (0,4) and (-1,1) respectively. With the aid of (3.12), from stated follows

Theorem 4. *1. Eigenvalues of modified GS operator (3.6), (3.3) in n- dimensional case, $n = 2, 3, \ldots$, are ($\nu = 1, 2, ..., [n/2]$):*

$$\lambda_\nu^{(1,2)}(k) = \frac{k}{k+1} + \frac{\mu_\nu ab}{2(k+1)^2}(1 \pm \sqrt{1 + \frac{4k(k+1)}{\mu_\nu ab}}). \tag{3.13}$$

In case of odd n there exists an additional root $\lambda_0 = k/(k+1)$. Here $\mu_\nu = 2(x_\nu + 1)$ and x_ν - the roots of Chebyshev polynomials of second kind which sets by formula (1.10).

2. Optimum parameter for convergence of series (4) with modified GS operator is

$$k_0 = \frac{1}{2}(\sqrt{1 - \mu_{max} ab} - 1), \tag{3.14}$$

where $\mu_{max} = 2(x_{max} + 1)$ and x_{max} - a maximal root of Chebyshev polynomials of second kind, which sets by formula (1.12).

3. Spectral radius of optimum modified GS operator is

$$\rho(\hat{B}_{k_0}) = \frac{|1 - \sqrt{1 - \mu_{max} ab}|^2}{\mu_{max}|ab|},$$

For $ab < 1/\mu_{max}$ and only for them optimum modified GS method is convergent.

From (3.13) follows, that at $k = 0$, that is for usual GS operator, the least eigenvalue is $\lambda_0 = 0$, and the greatest $\lambda_1 = \lambda_{max} = \mu_{max} ab$. Therefore convergence of a usual method takes place for $|ab| < 1/\mu_{max}$, that is significantly less then domain of convergence of the optimum modified method.

The matrix in (3.4) has same eigenvalues and, if to set the task of definition of optimum parameter for this matrix we shall receive, in the case of real ab,

$$k_0 = -\frac{\lambda_1 + \lambda_2}{2} = -\frac{\mu_{max} ab}{2},$$

that essentially differs from (3.14).

The parameter μ_{max} in (3.14) is in interval [1,4) that follows from (1.12). The left value takes place at $n = 2$, and right is a limit for the big matrices at $n \to \infty$.

In terms of matrix A (8) domain of convergency of modified GS method is

$$a_{-1} a_1 < \frac{a_0^2}{\mu_{max}}.$$

The same domain of convergence (2.3) has an optimum Richardson method, but an optimum GS method always converges faster. In fig. 4 is shown the behavior of $\rho(\hat{B}_{k_0})$ and $\rho(B_{k_0})$ for this two methods like functions of

$$x = \frac{\mu_{max} a_{-1} a_1}{a_0^2}.$$

In figure it accordingly $\rho S(x)$ and $\rho R(x)$. Here function of the attitude of spectral radii $y(x) = \rho S(x)/\rho R(x)$ is shown also.

The Modified GS Method described above is the Successive Overrelaxation Method (SOR)[1] in other designations $k = (1 - \omega)/\omega$, $\omega = 1/(k + 1)$. The result (3.14) for optimum parameter is in a good agreement with the theorem Kahan [5], which shows that SOR fails to converge if ω is outside the interval, i.e. if k is outside the interval $(-0.5, \infty)$. From (3.14) follows, that the maximal domain of convergence $ab < 1/\mu_{max}$ corresponds to interval $(-0.5, \infty)$.

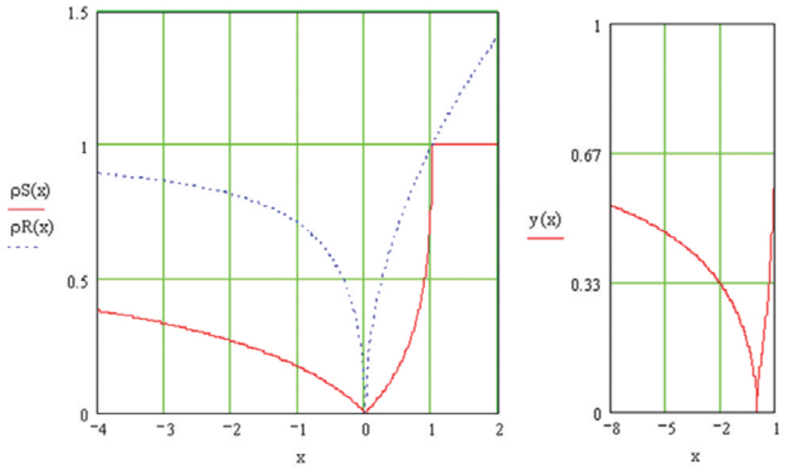

Fig. 4.

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