

Simulation of electromagnetic wave propagation in anisotropic media^{*}

Valery G. Yakhno¹, Tatyana Yakhno² and Mustafa Kasap¹

¹ Department of Mathematics, Faculty of Arts and Sciences, Dokuz Eylül University, Buca, 35160, Izmir, Turkey;

e-mail: valery.yakhno@deu.edu.tr,

e-mail: mustafa.kasap@cs.deu.edu.tr

² Computer Engineering Department, Engineering Faculty, Dokuz Eylül University, Bornova, Izmir, Turkey;

e-mail: yakhno@cs.deu.edu.tr

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Summary. Explicit formulas for fundamental and generalized solutions of the Cauchy problem for Maxwell's system are obtained for the case when the dielectric permeability is a symmetric positive definite matrix, the magnetic permeability is a positive constant, the conductivity vanished. The visualization of electromagnetic wave propagation made using these formulas by MatLAB, C++.

Key words: Maxwell's system, Cauchy problem, fundamental solution, modelling, MatLAB

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1. Introduction

The mathematical models of wave dynamic processes inside anisotropic electromagnetic bodies are described by Maxwell's system of electrodynamics [1, 2]. Electromagnetic properties of anisotropic electromagnetic bodies are given by the matrices of material parameters

$$\boldsymbol{\epsilon} = (\epsilon_{ij})_{3 \times 3}, \quad \boldsymbol{\mu} = (\mu_{ij})_{3 \times 3}, \quad \boldsymbol{\sigma} = (\sigma_{ij})_{3 \times 3},$$

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where ϵ, μ are positive definite; ϵ, μ are dielectric and magnetic probabilities, σ is the conductivity.

We note that isotropic materials are characterized by a special form of ϵ, μ, σ . This is the following

$$\epsilon = \epsilon I, \quad \mu = \mu I, \quad \sigma = \sigma I,$$

where $I = \text{diag}(1, 1, 1)$ is the unit matrix, $\epsilon > 0$, $\mu > 0$, σ are constants. All other forms of ϵ, μ, σ , which are different from the isotropic form, describe anisotropic materials. In this paper we study the electromagnetic wave propagation in anisotropic materials in which $\epsilon = (\epsilon_{ij})_{3 \times 3}$ is a symmetric positive definite matrix, $\mu = \mu I$, $\mu > 0$ is a constant, $\sigma = 0$.

The study of electromagnetic wave propagation in anisotropic materials is active research subject related to computer facilities, measurement technologies, geophysics and others [3].

We will use the following notations, laws and constitutive relations of electromagnetic theory:

$E = (E_1, E_2, E_3)$, $H = (H_1, H_2, H_3)$ are electric and magnetic intensity vectors, $E_j = E_j(x, t)$, $H_j = H_j(x, t)$, $j = 1, 2, 3$; $x = (x_1, x_2, x_3) \in \mathfrak{R}^3$, $t \in \mathfrak{R}$; D is the electric displacement; B is the magnetic induction; \mathbf{j} is the current density; ρ is the density of electric charges, c is the light velocity;

$$\frac{\partial \rho}{\partial t} + \text{div} \mathbf{j} = 0$$

is the law of the conservation of electric charge;

$$D = \epsilon E, \quad B = \mu H$$

are constitutive relations.

The complete set of Maxwell's equations has the form

$$\text{curl}_x H = \frac{1}{c} \frac{\partial D}{\partial t} + \frac{4\pi}{c} \mathbf{j}, \quad \text{curl}_x E = -\frac{1}{c} \frac{\partial B}{\partial t},$$

$$\text{div}_x B = 0, \quad \text{div}_x D = 4\pi \rho.$$

We assume in our paper that

$$E = 0, \quad H = 0, \quad \rho = 0, \quad \mathbf{j} = 0 \quad \text{for } t \leq 0.$$

This means there is no electromagnetic field, currents or electric charges at the time $t \leq 0$.

The Cauchy problem for Maxwell's system may be stated as follows. Find an electromagnetic vector field E, H satisfying

$$(1) \quad \operatorname{curl}_x H = \bar{\epsilon} \frac{\partial E}{\partial t} + \bar{\mathbf{j}}, \quad \operatorname{curl}_x E = -\bar{\mu} \frac{\partial H}{\partial t},$$

$$(2) \quad E|_{t \leq 0} = 0, \quad H|_{t \leq 0} = 0,$$

where

$$\bar{\epsilon} = \frac{\epsilon}{c}, \quad \bar{\mu} = \frac{\mu}{c}, \quad \bar{\mathbf{j}} = \left(\frac{4\pi}{c}\right)\mathbf{j}.$$

We note that ρ and $\bar{\mathbf{j}}$ are connected by

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} \bar{\mathbf{j}}.$$

Further we shall omit bars over letters $\epsilon, \mu, \mathbf{j}$ for simplicity of writing.

The section 2 of this paper is devoted to explicit formulas for fundamental and some generalized solutions of (1), (2). The section 3 deals with simulation of the wave propagation using these formulas. The section 4 contains some remarks and conclusions of our research.

2. Explicit formulas for generalized solutions of (1), (2)

Let $\epsilon = (\epsilon_{ij})_{3 \times 3}$ be a symmetric positive definite matrix with constant elements, μ be a positive number; $W = \epsilon^{-1}$ be the inverse matrix to ϵ ; $\mathbf{j} = \mathbf{e}\delta(x)\delta(t)$, where $\delta(t)$ is the Dirac delta function of one variable t , $\delta(x) = \delta(x_1)\delta(x_2)\delta(x_3)$, \mathbf{e} is an unit vector of \mathfrak{R}^3 .

2.1. The reduction of (1) to systems for vector and scalar electric potentials

We look for vector functions H and E in the form

$$(3) \quad H = \frac{1}{\mu} \operatorname{curl}_x A, \quad E = -\frac{\partial A}{\partial t} + \nabla_x \phi,$$

where A is a vector function and ϕ is a scalar function depending on $(x, t) \in \mathfrak{R}^4$. Substituting (3) into (1) we find

$$(4) \quad \epsilon \frac{\partial^2 A}{\partial t^2} - \frac{1}{\mu} \Delta_x A - \epsilon \frac{\partial}{\partial t} \nabla_x \phi + \frac{1}{\mu} \nabla_x \operatorname{div}_x A = \mathbf{j}.$$

Choosing the vector function $\Phi = \nabla\phi$ such that the following equalities hold

$$(5) \quad \frac{\partial}{\partial t}\Phi = K\nabla_x \operatorname{div}_x A, \quad \Phi|_{t \leq 0} = 0,$$

we obtain from (4)

$$(6) \quad \frac{\partial^2 A}{\partial t^2} - K\Delta A = W\mathbf{j}, \quad x \in \mathfrak{R}^3, \quad t \in \mathfrak{R}, \quad K = \frac{1}{\mu}W.$$

Further we will consider (6) subject to the initial data

$$(7) \quad A|_{t=0} = 0, \quad \left. \frac{\partial A}{\partial t} \right|_{t=0} = 0, \quad x \in \mathfrak{R}^3.$$

2.2. Explicit formula for a solution of (6),(7)

Under assumptions of the section 2, the matrix K is a real symmetric positive definite one. According to the matrix theory, K is congruent to a diagonal matrix of its eigenvalues. This means that there exists an orthogonal matrix $T = (T_{ij})_{3 \times 3}$ such that

$$T^{-1}KT = \Lambda, \quad T^{-1}WT = \mu\Lambda,$$

where $\Lambda = \operatorname{diag}(a_1^2, a_2^2, a_3^2)$, $T^{-1} = (T_{ij}^{-1})_{3 \times 3}$ is the inverse matrix to T .

The solution of the problem (6), (7) we seek in the form

$$(8) \quad A = T^{-1}Y,$$

where Y is an unknown function. Multiplying equations (6), (7) by T from the left hand side and then substituting (8) and $\mathbf{j} = \mathbf{e}\delta(x)\delta(t)$ into obtained equation we find

$$(9) \quad \frac{\partial^2 Y}{\partial t^2} - \Lambda\Delta_x Y = TW\mathbf{e}\delta(x)\delta(t), \quad x \in \mathfrak{R}^3, \quad t > 0,$$

$$(10) \quad Y(x, 0) = 0, \quad \left. \frac{\partial Y}{\partial t}(x, t) \right|_{t=0} = 0.$$

The solution of (9), (10) is given by

$$(11) \quad Y = (Y_1, Y_2, Y_3)^*,$$

$$Y_m = \frac{\theta(t)}{4\pi a^2|x|} (TW\mathbf{e})_m \delta\left(t - \frac{|x|}{a_m}\right), \quad m = 1, 2, 3,$$

where $(TW\mathbf{e})_m$ is m-th component of the vector $TW\mathbf{e}$, $*$ is the sign of transposition, $\theta(t)$ is the Heaviside step function.

Using (8), (11) we find the following explicit formula for a solution of (6), (7)

$$(12) \quad A = (A_1, A_2, A_3),$$

$$A_j = \frac{\theta(t)}{4\pi} \sum_{m=1}^3 \frac{1}{a_m^2} T_{jm}^{-1}(TW\mathbf{e})_m \frac{1}{|x|} \delta\left(t - \frac{|x|}{a_m}\right), \quad j = 1, 2, 3.$$

2.3. Explicit formula for a solution of (5)

The solution of (5) is given by the following convolution

$$(13) \quad \Phi(x, t) = K \int_{-\infty}^{+\infty} \theta(t - \tau) \nabla_x \operatorname{div}_x A(x, \tau) d\tau.$$

Using (11), (8) the formula (13) may be written as

$$(14) \quad \Phi = (\Phi_1, \Phi_2, \Phi_3),$$

$$\Phi_n = -\frac{\theta(t)}{4\pi} \sum_{i=1}^3 \sum_{j=1}^3 \sum_{m=1}^3 \frac{1}{a_m^2} K_{ni} T_{jm}^{-1}(TW\mathbf{e})_m$$

$$\times \left\{ \frac{\partial}{\partial x_i} \left(\frac{x_j}{|x|^3} \right) \theta\left(t - \frac{|x|}{a_m}\right) + \frac{1}{a_m} \left[\frac{\partial}{\partial x_i} \left(\frac{x_j}{|x|^2} \right) - \frac{x_j x_i}{|x|^4} \right] \right.$$

$$\left. \times \delta\left(t - \frac{|x|}{a_m}\right) - \frac{x_j x_i}{a_m^2 |x|^3} \delta'\left(t - \frac{|x|}{a_m}\right) \right\}, \quad n = 1, 2, 3;$$

where K_{ni} , $n, i = 1, 2, 3$ are elements of K .

2.4. Explicit formula for a solution of (1), (2) for $\mathbf{j} = \mathbf{e}\delta(x)\delta(t)$

Using (3), (12), (14) we find a solution of (1), (2) in the form

$$(15) \quad H = \frac{1}{\mu} \operatorname{curl}_x A, \quad E = -\frac{\partial A}{\partial t} + \Phi,$$

where

$$(16) \quad \frac{\partial A_j}{\partial x_i} = \frac{\theta(t)}{4\pi} \sum_{m=1}^3 \frac{1}{a_m^2} T_{jm}^{-1}(TW\mathbf{e})_m$$

$$\times \left[\frac{\partial}{\partial x_i} \left(\frac{1}{|x|} \right) \delta\left(t - \frac{|x|}{a_m}\right) - \frac{x_i}{|x|^2 a_m} \delta'\left(t - \frac{|x|}{a_m}\right) \right], \quad i, j = 1, 2, 3;$$

$$(17) \quad \frac{\partial A_j}{\partial t} = \frac{\theta(t)}{4\pi} \sum_{m=1}^3 \frac{1}{a_m^2} T_{jm}^{-1}(TW\mathbf{e})_m \frac{1}{|x|} \delta' \left(t - \frac{|x|}{a_m} \right),$$

$j = 1, 2, 3$; Φ is defined by (14).

2.5. *Explicit formulas for a solution of (1), (2) for $\mathbf{j} = \mathbf{e}\delta(x)f(t)$.*

Let $\mathbf{j} = \mathbf{e}\delta(x)f(t)$, $f(t) \in C^1(t \geq 0)$, $f(t)|_{t < 0} \equiv 0$. Then a solution of (1), (2) is given by the formula (15) in which

$$(18) \quad \begin{aligned} \frac{\partial A_j}{\partial x_i} &= -\frac{\theta(t)}{4\pi} \sum_{m=1}^3 \frac{1}{a_m^2} T_{jm}^{-1}(TW\mathbf{e})_m \frac{x_i}{|x|^2} \\ &\times \left\{ \frac{1}{a_m} [f]_{t=0} \delta \left(t - \frac{|x|}{a_m} \right) + \frac{1}{|x|} f \left(t - \frac{|x|}{a_m} \right) \right. \\ &\left. + \frac{1}{a_m} \theta \left(t - \frac{|x|}{a_m} \right) f' \left(t - \frac{|x|}{a_m} \right) \right\}, \quad i, j = 1, 2, 3; \end{aligned}$$

$$(19) \quad \begin{aligned} \frac{\partial A_j}{\partial t} &= \frac{\theta(t)}{4\pi} \sum_{m=1}^3 \frac{1}{a_m^2} T_{jm}^{-1}(TW\mathbf{e})_m \frac{1}{|x|} \\ &\times \left\{ [f]_{t=0} \delta \left(t - \frac{|x|}{a_m} \right) + \theta \left(t - \frac{|x|}{a_m} \right) f' \left(t - \frac{|x|}{a_m} \right) \right\}, \quad j = 1, 2, 3; \end{aligned}$$

$$(20) \quad \begin{aligned} \Phi_n &= -\frac{\theta(t)}{4\pi} \sum_{i=1}^3 \sum_{j=1}^3 \sum_{m=1}^3 \frac{1}{a_m^2} K_{ni} T_{jm}^{-1}(TW\mathbf{e})_m \\ &\times \left\{ \frac{\partial}{\partial x_i} \left(\frac{x_j}{|x|^3} \right) \theta \left(t - \frac{|x|}{a_m} \right) \int_0^{t - \frac{|x|}{a_m}} f(z) dz \right. \\ &+ \frac{1}{a_m} \left(\frac{\partial}{\partial x_i} \left(\frac{x_j}{|x|^2} \right) - \frac{x_i x_j}{|x|^4} \right) f \left(t - \frac{|x|}{a_m} \right) - \frac{x_j x_i}{a_m^2 |x|^3} \left(\theta \left(t - \frac{|x|}{a_m} \right) \right. \\ &\left. \left. \times f' \left(t - \frac{|x|}{a_m} \right) + [f]_{t=0} \delta \left(t - \frac{|x|}{a_m} \right) \right) \right\}, \\ &n = 1, 2, 3, \quad [f]_{t=0} = f(t+0) - f(t-0). \end{aligned}$$

2.6. Explicit formulas of electromagnetic field for isotropic medium

Let $\mathbf{j} = \mathbf{e}\delta(x)\delta(t)$, $\boldsymbol{\epsilon} = \epsilon I$, $\boldsymbol{\mu} = \mu I$, $\boldsymbol{\sigma} = 0$.

Then a solution of (1), (2) may be presented in the form

$$(21) \quad \begin{aligned} \mathbf{E}(x, t) = & \frac{\theta(t)}{4\pi a^2 |x| \epsilon} \left[\left(\frac{x}{|x|} \mathbf{e} \right) \frac{x}{|x|} - \mathbf{e} \right] \delta' \left(t - \frac{|x|}{a} \right) \\ & - \frac{\theta(t)}{4\pi a \epsilon} \left[\operatorname{div}_x \left(\frac{\mathbf{e}}{|x|} \right) \frac{x}{|x|} + \nabla_x \left(\frac{x}{|x|^2} \mathbf{e} \right) \right] \delta \left(t - \frac{|x|}{a} \right) \\ & + \frac{1}{4\pi \epsilon} \nabla_x \operatorname{div}_x \left(\frac{\mathbf{e}}{|x|} \right) \theta \left(t - \frac{|x|}{a} \right), \end{aligned}$$

$$(22) \quad \mathbf{H}(x, t) = \frac{1}{4\pi} \operatorname{curl}_x \left[\frac{\mathbf{e}}{|x|} \delta \left(t - \frac{|x|}{a} \right) \right].$$

3. Visualization

The tools that we use for a simulation of electromagnetic waves are described in this section. Images of electromagnetic fields visualization are given as a result.

There are two types of images on the figures. The first type is related to three dimensional pictures of wave shapes. Wave shapes viewed over a rectangular region for fixed time and one fixed space variable x_3 . The second type deals with isolines of wave shapes within two space variables for fixed time. The density of the color of the picks in the z -axis realizes the magnitude of wave fields. Precisely the dark (blue) is related to negative and light (red) corresponds to positive values.

3.1. 3-D Graphs

According to the formulas above, we generate the graphs of several components of E and H components. The z axis represents one of E, H components, other axes represents x_1 and x_2 variables, respectively. We fixed $x_3 = 1$ and z axis has the logarithmic 10 scale. The parameters of media in considered examples are the following. The matrix $\boldsymbol{\epsilon}$ for the anisotropic medium is given by

$$\begin{pmatrix} 0.6873 & 0.1556 & 0.8560 \\ 0.3461 & 0.1911 & 0.4902 \\ 0.1660 & 0.4225 & 0.8159 \end{pmatrix}$$

and $\epsilon = 1$ for the isotropic medium. For all examples $\mu = 2$.

The figures (1a), (1b), (3a), (2b) are related to the source $\mathbf{j} = \mathbf{e}\delta(x)\delta(t)$, and (3a), (3b), (4) are connected to $\mathbf{j} = \mathbf{e}\delta(x)f(t)$, $f(t) = \theta(t)\sin\omega t$, $e = (1, 1, 1)$.

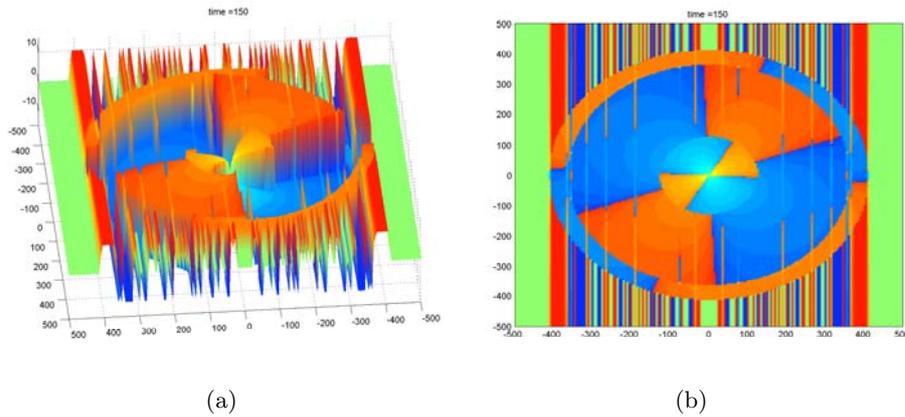


Fig. 1. Anisotropic medium; Component E_3 .

Three dimensional simulation of E_3 is represented on Fig.(1a). In Fig.(1b) we plot the value of E_3 as rectangular array of cells with colors on a surface. In both pictures we can see three distinctive wave fronts propagating through the point source.

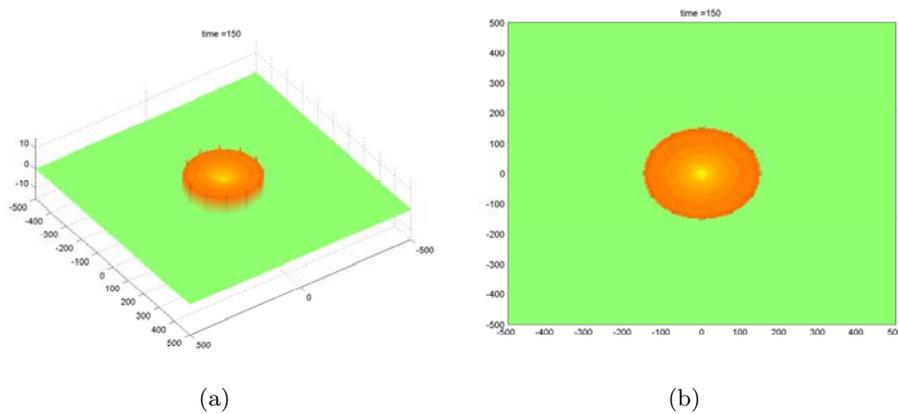


Fig. 2. Isotropic medium; Component E_3 .

In Fig.2 there is only one wave front observed as it is expected in isotropic media. The simulation of the first component of the electric field in the anisotropic medium by the source $\mathbf{j} = \theta(t)\mathbf{e}\delta(x) \sin \omega t$ for $\omega = 0.05$ and $\omega = 10$ is presented on Fig.(3a) and Fig.(3b).

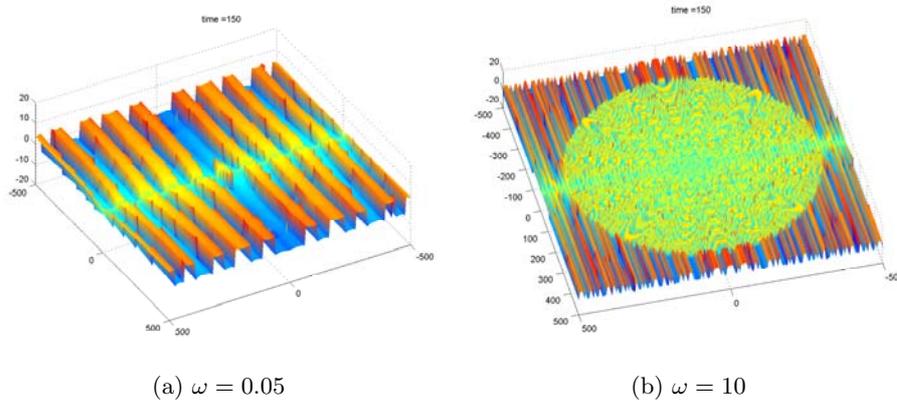


Fig. 3. Anisotropic medium; Component E_1 ; $f(t) = \theta(t) \sin \omega t$

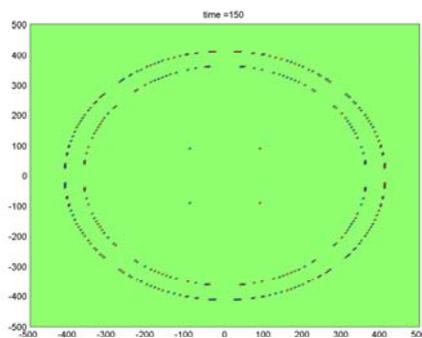


Fig. 4. Anisotropic medium; Component H_1 ; $f(t) = \theta(t) \sin 10t$

The figure 4 is related to the first component of the magnetic field in anisotropic medium with the source $\mathbf{j} = \theta(t)\mathbf{e}\delta(x) \sin 10t$.

3.2. Fast running animations

Generating animation of the propagation is time consuming operation. First of all in a limited time range we simulate the propagation for a specific time value. At the end we collect this frames and show them continuously. To overcome time constraint we use special tools to make operations faster. By means of MatLAB we wrote readable code which is easy for modifications. Again with MatLAB we translate this code in to fast running programming language C++. At the end we use Intel's processor specific C++ compiler to generate binary code for the animations.

4. Conclusions

Electromagnetic fields arising from the point sources were simulated by explicit formulas for generalized solutions of the Cauchy problem for Maxwell's system for the following case of the anisotropy. The electric permeability is given by a symmetric positive definite matrix, magnetic permeability is a positive constant, the conductivity vanished.

We note that the simulation of electromagnetic fields based on explicit formulas is the best one. But it is impossible to find explicit formulas for solutions in general case. For this we need to construct approximate solutions by numerical procedures and methods, and then simulate electromagnetic fields of these solutions.

The results of our paper may be used to test simulation of electromagnetic waves in this general situation.

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