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On properties of a class of matrix differential operators in $\boldsymbol{R}^n$ ⋆

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Summary. In the paper we consider a class of matrix quasi-elliptic operators in R^n. We establish isomorphic properties of these operators in special weighted Sobolev spaces $W^l_{p,\sigma}(R^n)$. From our results a theorem on isomorphism for the Navier–Stokes operator follows.

Key words: matrix quasi-elliptic operators, the Navier–Stokes operator, isomorphism, weighted Sobolev spaces

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1. Introduction

The paper is devoted to the study of a class of matrix quasi-elliptic operators:

$$\mathcal{L}(D_x) = \begin{pmatrix} K(D_x) & L(D_x) \\ M(D_x) & 0 \end{pmatrix} \tag{1.1}$$

in the whole space R^n. In particular, the class includes the Navier–Stokes operator

$$\ell(D_x) = \begin{pmatrix} -\Delta & 0 & 0 & D_{x_1} \\ 0 & -\Delta & 0 & D_{x_2} \\ 0 & 0 & -\Delta & D_{x_3} \\ D_{x_1} & D_{x_2} & D_{x_3} & 0 \end{pmatrix}, \quad x \in R^3. \tag{1.2}$$

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Our aim is to prove a theorem on isomorphic properties of the operators (1.1).

To prove theorems on isomorphic properties of differential operators, as a rule, it is necessary to use special weighted Sobolev spaces. In the paper we use the class of special weighted Sobolev spaces $W^l_{p,\sigma}(R^n)$ which was defined by the author in [1].

In the paper we continue the investigations [2–4].

2. Statement of the main result

Consider the matrix operator (1.1). We will suppose that the $\nu \times \nu$-operator $\mathcal{L}(D_x)$ satisfies the following conditions.

Condition 1. The operators $K(D_x)$, $L(D_x)$ and $M(D_x)$ are $\mu \times \mu$, $\mu \times (\nu - \mu)$ and $(\nu - \mu) \times \mu$ matrix differential operators in x respectively.

Condition 2. There exists a vector $\alpha = (\alpha_1, \dots, \alpha_n)$, $1/(2\alpha_i) \in N$, such that for any $c > 0$ the equalities hold

$$K(c^\alpha i\xi) = c\, K(i\xi), \quad L(c^\alpha i\xi) = c^{1/2}\, L(i\xi), \quad M(c^\alpha i\xi) = c^{1/2}\, M(i\xi),$$

i. e. the symbols of the matrix operators $K(D_x)$, $L(D_x)$ and $M(D_x)$ are homogeneous with respect to the vector α.

Condition 3. The equality

$$\det K(i\xi) = 0, \quad \xi \in R^n,$$

holds if and only if $\xi = 0$.

Condition 4. The inequality holds

$$\det(M(i\xi)K^{-1}(i\xi)L(i\xi)) \neq 0, \quad \xi \in R^n.$$

Remark 1. From conditions 2 and 3 it follows that the operator $K(D_x)$ is a quasi-elliptic operator (see, for example, [5]).

As an example we consider the Navier–Stokes operator (1.2). Note, that this operator is by Douglis–Nirenberg elliptic operator. For it we, obviously, have $\nu = 4$, $\mu = 3$,

$$K(D_x) = \begin{pmatrix} -\Delta & 0 & 0 \\ 0 & -\Delta & 0 \\ 0 & 0 & -\Delta \end{pmatrix}, \quad L(D_x) = \begin{pmatrix} D_{x_1} \\ D_{x_2} \\ D_{x_3} \end{pmatrix},$$

$$M(D_x) = (D_{x_1}, D_{x_2}, D_{x_3}),$$

$$\det(M(i\xi)K^{-1}(i\xi)L(i\xi)) = -1, \quad \alpha = (1/2, 1/2, 1/2).$$

Hence, Conditions 1–4 for the Navier–Stokes operator are satisfied.

Using our results [1–4], we can establish analogous assertions for the operator (1.1) in special weighted Sobolev spaces $W^l_{p,\sigma}(R^n)$.

By definition [1], the norm in the weighted Sobolev spaces

$$W^l_{p,\sigma}(R^n), \quad l=(l_1,\dots,l_n), \quad l_i\in N, \quad 1<p<\infty, \quad \sigma\ge 0,$$

is defined by

$$\|U(x),\, W^l_{p,\sigma}(R^n)\| = \sum_{0\le\beta/l\le 1} \|(1+\langle x\rangle)^{-\sigma(1-\beta/l)} D^\beta_x U(x),\, L_p(R^n)\|, \tag{2.1}$$

where

$$\langle x\rangle^2=\sum_{i=1}^n x_i^{2l_i}, \quad \beta=(\beta_1,\dots,\beta_n), \quad \beta/l=\sum_{i=1}^n \beta_i/l_i.$$

Remark 2. In particular, $W^l_{p,0}(R^n)$ is the Sobolev space $W^l_p(R^n)$.

Remark 3. Consider the isotropic case $l_1=\dots=l_n=l$ with $\sigma=1$. Obviously, the norm (2.1) is equivalent to

$$\sum_{0\le|\beta|\le l} \|(1+|x|)^{-\sigma(l-|\beta|)} D^\beta_x U(x),\, L_p(R^n)\|. \tag{2.2}$$

If $p>n$, spaces with the norm (2.2) were introduced by L. D. Kudryavtsev [6] (see also the survey [7]). For every $p>1$ these spaces were considered by L. Nirenberg, H. F. Walker, M. Cantor (see, for example, [8, 9]).

By $\overset{\circ}{W}{}^l_{p,\sigma}(R^n)$ we denote the completion $C^\infty_0(R^n)$ with respect to the norm (2.1).

From the definitions it follows that the space $\overset{\circ}{W}{}^l_{p,\sigma}(R^n)$ is embedded in the space $W^l_{p,\sigma}(R^n)$. One can show that, for sufficiently large σ, the strict embedding holds

$$\overset{\circ}{W}{}^l_{p,\sigma}(R^n)\subset W^l_{p,\sigma}(R^n).$$

As was proven in [1] the equality holds

$$\overset{\circ}{W}{}^l_{p,\sigma}(R^n)= W^l_{p,\sigma}(R^n)$$

for $\sigma\le 1$. Henceforth we consider these spaces for $\sigma=1$.

Theorem. *Let*

$$l = (l_1, \dots, l_n) = (1/\alpha_1, \dots, 1/\alpha_n), \quad l/2 = (l_1/2, \dots, l_n/2).$$

Suppose that $\sum_1^n \alpha_i/p > 1$. *Then the operator (1.1)*

$$\mathcal{L}(D_x) : \prod_1^{\mu} W_{p,1}^{l}(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n) \to \prod_1^{\mu} L_p(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$$

is an isomorphism.

Corollary. *For* $1 < p < 3/2$ *the Navier–Stokes operator (1.2)*

$$\ell(D_x) : \prod_1^{3} W_{p,1}^{2}(R^3) \times W_{p,1}^{1}(R^3) \to \prod_1^{3} L_p(R^3) \times W_{p,1}^{1}(R^3)$$

is an isomorphism.

Remark 4. Some results on isomorphic properties of differential operators with homogeneous symbols in R^n are contained in [2–4, 10–14].

3. The scheme of the proof of the main result

In this section we present the scheme of the proof of the isomorphic properties of the operator (1.1).

In the beginning we show that the operator $\mathcal{L}(D_x)$ takes the space

$$\prod_1^{\mu} W_{p,1}^{l}(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$$

into the space

$$\prod_1^{\mu} L_p(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n).$$

By the definition (1.1), for every vector-function

$$u(x) = (u^+(x), u^-(x)) \in \prod_1^{\mu} W_{p,1}^{l}(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$$

we have $f(x) = \mathcal{L}(D_x)u(x)$, where $f(x) = (f^+(x), f^-(x))$:

$$f^+(x) = K(D_x)u^+(x) + L(D_x)u^-(x), \quad f^-(x) = M(D_x)u^+(x).$$

Since $u^+(x) \in \prod_1^\mu W^l_{p,1}(R^n)$ and $u^-(x) \in \prod_{\mu+1}^\nu W^{l/2}_{p,1}(R^n)$, then, by Condition 2, $f^+(x) \in \prod_1^\mu L_p(R^n)$. By analogy we have

$$(1+\langle x\rangle)^{-(1/2-\beta/l)} D_x^\beta M(D_x)u^+(x) \in \prod_{\mu+1}^{\nu} L_p(R_n), \quad 0 \le \beta/l \le 1/2,$$

which is equivalent to the fact that

$$(1+\langle x\rangle^{1/2})^{-(1-2\beta/l)} D_x^\beta M(D_x)u^+(x) \in \prod_{\mu+1}^{\nu} L_p(R_n), \quad 0 \le 2\beta/l \le 1.$$

Taking into account the estimate

$$c_1 \left(\sum_{i=1}^n x_i^{l_i}\right)^{1/2} \le \langle x\rangle^{1/2} \le c_2 \left(\sum_{i=1}^n x_i^{l_i}\right)^{1/2}, \quad 0 < c_1 < c_2,$$

by the definition of the space $W^{l/2}_{p,1}(R^n)$, $l/2 = (l_1/2, \dots, l_n/2)$, we obtain

$$f^-(x) = M(D_x)u^+(x) \in \prod_{\mu+1}^{\nu} W^{l/2}_{p,1}(R^n).$$

Consequently, the range of the operator $\mathcal{L}(D_x)$ lies in the space

$$\prod_1^\mu L_p(R^n) \times \prod_{\mu+1}^{\nu} W^{l/2}_{p,1}(R^n).$$

Therefore, to prove the theorem, it suffices to demonstrate that the system of differential equations in R^n:

$$\begin{aligned} K(D_x)u^+ + L(D_x)u^- &= f^+(x), \\ M(D_x)u^+ &= f^-(x) \end{aligned} \tag{3.1}$$

has a unique solution

$$u^+(x) \in \prod_1^\mu W^l_{p,\sigma}(R^n), \quad u^-(x) \in \prod_{\mu+1}^{\nu} W^{l/2}_{p,\sigma}(R^n), \tag{3.2}$$

for every right-hand side

$$f^+(x) \in \prod_1^\mu L_p(R^n), \quad f^-(x) \in \prod_{\mu+1}^{\nu} W^{l/2}_{p,1}(R^n), \tag{3.3}$$

and the estimate holds

$$(3.4)\qquad \sum_{j=1}^{\mu} \|u_j^+(x),\, W_{p,1}^{l}(R^n)\| + \sum_{i=\mu+1}^{\nu} \|u_i^-(x),\, W_{p,1}^{l/2}(R^n)\|$$
$$\leq c \left(\sum_{j=1}^{\mu} \|f_j^+(x),\, L_p(R^n)\| + \sum_{i=\mu+1}^{\nu} \|f_i^-(x),\, W_{p,1}^{l/2}(R^n)\| \right)$$

with a constant $c > 0$ independent of $f^+(x)$, $f^-(x)$.

In the next section we present the basic points of the proof of solvability of the system (3.1).

4. The proof of solvability of the system (3.1)

By analogy with [2–4], we present the scheme of the proof of solvability of (3.1).

First, we construct a sequence of approximate solutions to (3.1) by using the Uspenskii integral representation for summable functions (see [15, Chapter 1]):

$$(4.1)\qquad \varphi(x) = \lim_{h\to 0} (2\pi)^{-n} \int_h^{h^{-1}} v^{-|\alpha|-1}$$
$$\times \int_{R^n} \int_{R^n} \exp\left(i \frac{x-y}{v^{\alpha}} \xi \right) G(\xi) \varphi(y)\, d\xi dy dv,$$

where $|\alpha| = \sum_1^n \alpha_i$ and

$$(4.2)\qquad G(\xi) = 2m\langle \xi \rangle^{2m} \exp(-\langle \xi \rangle^{2m}), \quad \langle \xi \rangle^2 = \sum_{i=1}^{n} \xi_i^{2/\alpha_i}, \quad m \in N.$$

For this we consider the following system with a parameter $\xi \in R^n$:

$$K(i\xi)v^+ + L(i\xi)v^- = \widehat{f}^+(\xi), \quad M(i\xi)v^+ = \widehat{f}^-(\xi).$$

Note, the system is obtained by formal application of the Fourier operator to (3.1). Taking into account Conditions 1–4, for $\xi \in R^n \backslash \{0\}$, we obviously obtain

$$(4.3)\qquad v^+(\xi) = K^{-1}(i\xi)(I - L(i\xi)N_0^{-1}(\xi)M(i\xi)K^{-1}(i\xi))\widehat{f}^+(\xi)$$
$$+K^{-1}(i\xi)L(i\xi)N_0^{-1}(\xi)\widehat{f}^-(\xi),$$

(4.4) $$v^-(\xi) = N_0^{-1}(\xi)M(i\xi)K^{-1}(i\xi)\widehat{f}^+(\xi) - N_0^{-1}(\xi)\widehat{f}^-(\xi),$$

where

$$N_0(\xi) = M(i\xi)K^{-1}(i\xi)L(i\xi).$$

By analogy with [15, Chapter 3], we construct the vector-functions

(4.5) $$u_k^+(x) = (2\pi)^{-n/2} \int_{1/k}^{k} v^{-1} \int_{R^n} e^{ix\xi} G(\xi v^\alpha) v^+(\xi)\, d\xi\, dv,$$

(4.6) $$u_k^-(x) = (2\pi)^{-n/2} \int_{1/k}^{k} v^{-1} \int_{R^n} e^{ix\xi} G(\xi v^\alpha) v^-(\xi)\, d\xi\, dv.$$

From definitions $v^+(\xi)$, $v^-(\xi)$ it follows that

$$K(D_x)u_k^+(x) + L(D_x)u_k^-(x) = f_k^+(x),$$

$$M(D_x)u_k^+(x) = f_k^-(x),$$

where

$$f_k^+(x) = (2\pi)^{-n} \int_{1/k}^{k} v^{-|\alpha|-1} \int_{R^n}\int_{R^n} \exp\left(i\frac{x-y}{v^\alpha}\xi\right) G(\xi) f^+(y)\, d\xi\, dy\, dv,$$

$$f_k^-(x) = (2\pi)^{-n} \int_{1/k}^{k} v^{-|\alpha|-1} \int_{R^n}\int_{R^n} \exp\left(i\frac{x-y}{v^\alpha}\xi\right) G(\xi) f^-(y)\, d\xi\, dy\, dv.$$

By the integral representation (4.1), we have

$$\|f_k^+(x) - f^+(x),\, L_p(R_n)\| \to 0, \quad k \to \infty,$$

$$\|(1 + \langle x\rangle^{1/2})^{-(1-2\beta/l)}(D_x^\beta f_k^-(x) - D_x^\beta f^-(x)),\, L_p(R_n)\| \to 0,$$

$$0 \le 2\beta/l \le 1, \quad k \to \infty.$$

Consequently, we can consider the vector-functions (4.5), (4.6) as an approximate solution to (3.1).

Suppose that the vector-functions $f^+(x)$, $f^-(x)$ are compactly-supported. By (4.1), (4.2), one can indicate a natural number m such that the vector-functions $u_k^+(x)$, $u_k^-(x)$ are infinitely differentiable and summable with an arbitrary power $p \ge 1$ (see, for example, [4]).

Rewrite (4.5), (4.6) in operator form

$$(u_k^+(x),\, u_k^-(x)) = P_k f(x), \quad f(x) = (f^+(x),\, f^-(x)).$$

By analogy with [2–4], in the case of $|\alpha|/p > 1$ one can prove the estimate

$$(4.7)\qquad \sum_{j=1}^{\mu} \|u_{k,j}^{+}(x),\, W_{p,1}^{l}(R^n)\| + \sum_{i=\mu+1}^{\nu} \|u_{k,i}^{-}(x),\, W_{p,1}^{l/2}(R^n)\|$$

$$\le c\left(\sum_{j=1}^{\mu} \|f_j^{+}(x),\, L_p(R^n)\| + \sum_{i=\mu+1}^{\nu} \|f_i^{-}(x),\, W_{p,1}^{l/2}(R^n)\|\right)$$

with a constant $c > 0$ independent of $f^{+}(x)$, $f^{-}(x)$, k, and also establish the convergence

$$(4.8)\qquad \sum_{j=1}^{\mu} \|u_{k_1,j}^{+}(x) - u_{k_2,j}^{+}(x),\, W_{p,1}^{l}(R^n)\|$$

$$+ \sum_{i=\mu+1}^{\nu} \|u_{k_1,i}^{-}(x) - u_{k_2,i}^{-}(x),\, W_{p,1}^{l/2}(R^n)\| \to 0, \quad k_1, k_2 \to \infty.$$

Since the space $\prod\limits_{1}^{\mu} W_{p,1}^{l}(R^n) \times \prod\limits_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$ is complete, then one can construct a continuous linear operator

$$P : \prod_{1}^{\mu} L_p(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n) \;\to\; \prod_{1}^{\mu} W_{p,1}^{l}(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n),$$

that is defined on compactly-supported vector-functions by the formula:

$$Pf(x) = \lim_{k\to\infty} P_k f(x), \quad f(x) = (f^{+}(x),\, f^{-}(x)).$$

Obviously, the vector-function

$$u(x) = (u^{+}(x),\, u^{-}(x)) = Pf(x)$$

is a solution to (3.1). By denseness of the set compactly-supported vector-functions in $\prod\limits_{1}^{\mu} L_p(R^n) \times \prod\limits_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$ (see [1]) and the theorem on extension by continuity, we can extend the operator P to the whole space $\prod\limits_{1}^{\mu} L_p(R^n) \times \prod\limits_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$ with the same norm. We will use the same notation P for the extended operator.

From (4.7) it follows that the linear operators

$$P_k : \prod_{1}^{\mu} L_p(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n) \;\to\; \prod_{1}^{\mu} W_{p,1}^{l}(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$$

are continuous and the sequence $\{\|P_k\|\}$ is bounded, i. e. $\|P_k\| \leq c$. Consequently, by the Banach–Steinhaus theorem the convergence

$$P_k f(x) \to P f(x), \quad k \to \infty,$$

is true for every vector-function $f(x)$:

$$f^+(x) \in \prod_1^{\mu} L_p(R^n), \quad f^-(x) \in \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n).$$

The above implies existence of a solution $u(x)$:

$$u^+(x) \in \prod_1^{\mu} W_{p,1}^{l}(R^n), \quad u^-(x) \in \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$$

to the system (3.1) for every right-hand side $f(x)$:

$$f^+(x) \in \prod_1^{\mu} L_p(R^n), \quad f^-(x) \in \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n),$$

and the solution satisfies the estimate (3.4).

By analogy with [2, 10], one can prove uniqueness of a solution (3.2).

Consequently, the linear operator

$$\mathcal{L}(D_x) : \prod_1^{\mu} W_{p,1}^{l}(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n) \to \prod_1^{\mu} L_p(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n)$$

is continuous, its range coincides with the whole space

$$\prod_1^{\mu} L_p(R^n) \times \prod_{\mu+1}^{\nu} W_{p,1}^{l/2}(R^n),$$

the kernel is zero, and the operator P is its inverse. Hence, the operator $\mathcal{L}(D_x)$ is an isomorphism.

Thus, to complete the proof of the theorem it is necessary to obtain the estimate (4.7) and establish the convergence (4.8). In the next section we discuss the questions.

5. The estimates of approximate solutions

By analogy with [2–4], we present the scheme of the proof of the estimate (4.7) for approximate solutions.

Rewrite the vector-functions (4.5) and (4.6) as follows

$$u_k^+(x) = u_k^{+,+}(x) + u_k^{+,-}(x)$$

$$= (2\pi)^{-n/2} \int_{1/k}^{k} v^{-1} \int_{R^n} e^{ix\xi} G(\xi v^\alpha)(v^{+,+}(\xi) + v^{+,-}(\xi))\, d\xi\, dv,$$

$$u_k^-(x) = u_k^{-,+}(x) + u_k^{-,-}(x)$$

$$= (2\pi)^{-n/2} \int_{1/k}^{k} v^{-1} \int_{R^n} e^{ix\xi} G(\xi v^\alpha)(v^{-,+}(\xi) + v^{-,-}(\xi))\, d\xi\, dv,$$

where

$$v^{+,+}(\xi) = K^{-1}(i\xi)(I - L(i\xi)N_0^{-1}(\xi)M(i\xi)K^{-1}(i\xi))\widehat{f}^+(\xi),$$

$$v^{+,-}(\xi) = K^{-1}(i\xi)L(i\xi)N_0^{-1}(\xi)\widehat{f}^-(\xi),$$

$$v^{-,+}(\xi) = N_0^{-1}(\xi)M(i\xi)K^{-1}(i\xi)\widehat{f}^+(\xi),$$

$$v^{-,-}(\xi) = -N_0^{-1}(\xi)\widehat{f}^-(\xi), \quad N_0(\xi) = M(i\xi)K^{-1}(i\xi)L(i\xi).$$

We divide into the proof of the estimate (4.7) and the convergence (4.8) on four lemmas.

First, we present estimates for the higher order derivatives of $u_k^+(x)$, $u_k^-(x)$.

Lemma 1. *Let* $\beta = (\beta_1, \dots, \beta_n)$, $\beta\alpha = 1$. *Then the estimates are true*

$$\sum_{j=1}^{\mu} \|D_x^\beta u_{k,j}^{+,+}(x),\, L_p(R^n)\| \le c \sum_{j=1}^{\mu} \|f_j^+(x),\, L_p(R^n)\|,$$

$$\sum_{j=1}^{\mu} \|D_x^\beta u_{k,j}^{+,-}(x),\, L_p(R^n)\| \le c \sum_{i=\mu+1}^{\nu} \sum_{\gamma\alpha=1/2} \|D_x^\gamma f_i^-(x),\, L_p(R^n)\|,$$

with some constant $c > 0$ *independent of* $f^+(x)$, $f^-(x)$ *and* k; *moreover*

$$\sum_{j=1}^{\mu} \|D_x^\beta u_{k_1,j}^+(x) - D_x^\beta u_{k_2,j}^+(x),\, L_p(R^n)\| \to 0, \quad k_1, k_2 \to \infty.$$

Lemma 2. *Let $\beta = (\beta_1, \dots, \beta_n)$, $\beta\alpha = 1/2$. Then the estimates hold*

$$\sum_{i=\mu+1}^{\nu} \|D_x^\beta u_{k,i}^{-,+}(x),\, L_p(R^n)\| \le c \sum_{j=1}^{\mu} \|f_j^+(x),\, L_p(R^n)\|,$$

$$\sum_{i=\mu+1}^{\nu} \|D_x^\beta u_{k,i}^{-,-}(x),\, L_p(R^n)\| \le c \sum_{i=\mu+1}^{\nu} \sum_{\gamma\alpha=1/2} \|D_x^\gamma f_i^-(x),\, L_p(R^n)\|,$$

with some constant $c > 0$ independent of $f^+(x)$, $f^-(x)$ and k; moreover

$$\sum_{i=\mu+1}^{\nu} \|D_x^\beta u_{k_1,i}^-(x) - D_x^\beta u_{k_2,i}^-(x),\, L_p(R^n)\| \to 0, \quad k_1, k_2 \to \infty.$$

Now we present estimates for the other derivatives of the vector-functions $u_k^+(x)$, $u_k^-(x)$.

Lemma 3. *Let $\beta = (\beta_1, \dots, \beta_n)$, $\beta\alpha < 1$ and $|\alpha|/p > 1$. Then the estimates are true*

$$\sum_{j=1}^{\mu} \|\langle x\rangle^{-(1-\beta\alpha)} D_x^\beta u_{k,j}^{+,+}(x),\, L_p(R^n)\| \le c \sum_{j=1}^{\mu} \|f_j^+(x),\, L_p(R^n)\|,$$

$$\sum_{j=1}^{\mu} \|\langle x\rangle^{-(1-\beta\alpha)} D_x^\beta u_{k,j}^{+,-}(x),\, L_p(R^n)\| \le c \sum_{i=\mu+1}^{\nu} \|f_i^-(x),\, W_{p,1}^{l/2}(R^n)\|,$$

with some constant $c > 0$ independent of $f^+(x)$, $f^-(x)$ and k; moreover

$$\sum_{j=1}^{\mu} \|(1 + \langle x\rangle)^{-(1-\beta\alpha)} (D_x^\beta u_{k_1,j}^+(x) - D_x^\beta u_{k_2,j}^+(x)),\, L_p(R^n)\| \to 0,$$

$$k_1, k_2 \to \infty.$$

Lemma 4. *Let $\beta = (\beta_1, \dots, \beta_n)$, $\beta\alpha < 1/2$ and $|\alpha|/p > 1$. Then the estimates hold*

$$\sum_{i=\mu+1}^{\nu} \|\langle x\rangle^{-(1/2-\beta\alpha)} D_x^\beta u_{k,i}^{-,+}(x),\, L_p(R^n)\| \le c \sum_{j=1}^{\mu} \|f_j^+(x),\, L_p(R^n)\|,$$

$$\sum_{i=\mu+1}^{\nu} \|\langle x\rangle^{-(1/2-\beta\alpha)} D_x^\beta u_{k,i}^{-,-}(x),\, L_p(R^n)\| \le c \sum_{i=\mu+1}^{\nu} \|f_i^-(x),\, W_{p,1}^{l/2}(R^n)\|,$$

with some constant $c > 0$ *independent of* $f^+(x)$, $f^-(x)$ *and* k; *moreover*

$$\sum_{i=\mu+1}^{\nu} \|(1+\langle x\rangle)^{-(1/2-\beta\alpha)}(D_x^\beta u_{k_1,i}^-(x) - D_x^\beta u_{k_2,i}^-(x)),\, L_p(R^n)\| \to 0,$$

$$k_1, k_2 \to \infty.$$

Repeating similar arguments of the proofs of Lemmas 3.1–3.5 in [4], we can prove the above lemmas.

Lemmas immediately yield estimate (4.7) and convergence (4.8) for approximate solutions (4.5), (4.6) to the system (3.1) which were used in Section 4 in the proof of the theorem.

6. Appendix

It is interesting to compare isomorphic properties of the Navier–Stokes operator (1.2) and isomorphic properties of the operator

$$\ell_1(D_x) = \begin{pmatrix} 1 & 0 & 0 & D_{x_1} \\ 0 & 1 & 0 & D_{x_2} \\ 0 & 0 & 1 & D_{x_3} \\ D_{x_1} & D_{x_2} & D_{x_3} & 0 \end{pmatrix}, \quad x \in R^3.$$

From [4] it follows that the operator

$$\ell_1(D_x) : \prod_1^3 W_{p,1}^1(R^3) \times W_{p,1}^2(R^3) \;\to\; \prod_1^3 W_{p,1}^1(R^3) \times L_p(R^3)$$

is an isomorphism for $1 < p < 3/2$. Hence, for every vector-functions

$$(u^+(x),\, u^-(x)) \in \prod_1^3 W_{p,1}^1(R^3) \times W_{p,1}^2(R^3)$$

the estimate is true

$$c_1 \Bigg(\|\frac{1}{(1+|x|)} u^+(x),\, L_p(R^3)\| + \|\nabla u^+(x),\, L_p(R^3)\|$$

$$+ \|\frac{1}{(1+|x|)^2} u^-(x),\, L_p(R^3)\|$$

$$+ \|\frac{1}{(1+|x|)} \nabla u^-(x),\, L_p(R^3)\| + \sum_{|\beta|=2} \|D_x^\beta u^-(x),\, L_p(R^3)\| \Bigg)$$

$$\le \|(u^+(x) \,+\, \nabla u^-(x)),\, L_p(R^3)\| + \|\operatorname{div} u^+(x),\, L_p(R^3)\|$$

$$\leq c_2 \Big(\| \frac{1}{(1+|x|)} u^+(x),\, L_p(R^3)\| + \|\nabla u^+(x),\, L_p(R^3)\|$$

$$+ \| \frac{1}{(1+|x|)^2} u^-(x),\, L_p(R^3)\|$$

$$+ \| \frac{1}{(1+|x|)} \nabla u^-(x),\, L_p(R^3)\| + \sum_{|\beta|=2} \|D_x^\beta u^-(x),\, L_p(R^3)\| \Big)$$

with constants $c_1,\ c_2 > 0$ independent of $u^+(x),\, u^-(x)$.

Note that, by the corollary from the theorem, for every vector-functions

$$(u^+(x),\, u^-(x)) \in \prod_1^3 W^2_{p,1}(R^3) \times W^1_{p,1}(R^3), \quad 1 < p < 3/2,$$

the following estimate holds

$$c_1 \Big(\| \frac{1}{(1+|x|)^2} u^+(x),\, L_p(R^3)\|$$

$$+ \| \frac{1}{(1+|x|)} \nabla u^+(x),\, L_p(R^3)\| + \sum_{|\beta|=2} \|D_x^\beta u^+(x),\, L_p(R^3)\|$$

$$+ \| \frac{1}{(1+|x|)} u^-(x),\, L_p(R^3)\| + \|\nabla u^-(x),\, L_p(R^3)\| \Big)$$

$$\leq \|(-\varDelta\, u^+(x) \ + \ \nabla u^-(x)),\, L_p(R^3)\| + \|\mathrm{div}\, u^+(x),\, L_p(R^3)\|$$

$$\leq c_2 \Big(\| \frac{1}{(1+|x|)^2} u^+(x),\, L_p(R^3)\|$$

$$+ \| \frac{1}{(1+|x|)} \nabla u^+(x),\, L_p(R^3)\| + \sum_{|\beta|=2} \|D_x^\beta u^+(x),\, L_p(R^3)\|$$

$$+ \| \frac{1}{(1+|x|)} u^-(x),\, L_p(R^3)\| + \|\nabla u^-(x),\, L_p(R^3)\| \Big)$$

with constants $c_1,\ c_2 > 0$ independent of $u^+(x),\, u^-(x)$.

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