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General solution of the Cauchy problem for the acoustic equation in the form of dynamic ray expansion

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Summary. Main object of this paper is the acoustic equation. Coefficients of this equation are smooth functions of the space variables. Existence and uniqueness theorem of the generalized solution of the Cauchy problem in the form of the dynamic and classical ray series is given.

Key words: Cauchy problem, generalized solution, acoustic equation, ray series expansion

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1. Introduction

Construction of a solution by classical series expansion for Helmholtz (wave) and hyperbolic equations was studied by V. M. Babich [1]. The fundamental solutions for the second order hyperbolic equations in the form of Hadamard series expansion were studied by V. M. Babich [1,2], V. G. Romanov [3]. We note that different forms of the presentation for generalized solutions of hyperbolic equations have been applied to study the modern problems of mathematical physics and applied mathematics (see, for instance, works [1,3,4]). The main object of the present paper is the dynamic ray series expansion for a generalized solution of the Cauchy problem for the acoustic equation.

Let x, t be variables,

$$x = (x_1, x_2, x_3) \in R^3, t \in R, x^0 = (x_1^0, x_2^0, x_3^0) \in R^3$$

be a parameter. Consider the following acoustic equation

$$u_{tt} = v^2(x)\Delta u - v^2(x)\nabla(\ln m(x))\nabla u + f(x, x^0, t). \tag{1}$$

Here $u(x, x^0, t)$ is a preasure of the acoustic medium at the point x and at the moment t ; $v(x)$ describes the velocity of the wave, $v(x) > 0$; $m(x)$ is the density of the acoustic media x, $m(x) > 0$; $f(x, x^0, t)$ is a given density of externally acting forces; Δ is the Laplace operator with respect to the variable x; ∇ is the gradient operator relative to the variable x. The equation (1) we will consider with the initial condition

$$u(x, x^0, t) \mid_{t<0} = 0, \tag{2}$$

and suppose that $f(x, x^0, t) = \delta(x - x^0, t)$ which means a pulse point source concentrating at the point $x = x_0$ and at the moment $t = 0$. From mathematical point of view $\delta(x - x^0, t)$ is the Dirac delta function with the support at the points $x = x_0$, $t = 0$. A generalized function $u(x, x^0, t)$ which satisfies (1), (2) is called a generalized (fundamental) solution of the Cauchy problem (1),(2).

Let $\tau(x, x^0)$ be the time required for the signal to get from the point x^0 to the point x . This function $\tau(x, x^0)$ satisfies the eikonal equation

$$|\nabla\tau(x, x^0)|^2 = \frac{1}{v^2(x)}$$

and the condition

$$|\tau(x, x^0)|_{x\to x^0} = 0.$$

The equation $\tau(x, x^0) = t$ defines the wave front from the point source at x^0 at the time t. The surface $\tau(x, x^0) = t$ is the characteristic conoid. The method of constructing characteristic conoid $\tau(x, x^0) = t$ consists in constructing separate lines, called bicharacteristics, that lie on the conoid and jointly form it. Projection of the bicharacteristic onto the space of x is called a ray. The rays are orthogonal to the surfaces $\tau(x, x^0) = t$. For finding rays we need to solve Euler's system (see, for instance, [3]).

In this paper x^0 is a fixed point and the following assumptions hold.

(A1) *functions $v(x)$, $m(x)$ have constant values in*

$$U_\varepsilon = \{x \in R^3, \left|x - x^0\right| \le \varepsilon\},$$

where ε is fixed positive small number, that is ,$v(x) = v_0$, $m(x) = m_0$ for $x \in U_\varepsilon$;

(A2)*functions $v(x)$, $m(x)$ are smooth (infinitely differentiable) with positive values ;*

(A3)*the family of the rays $\{T(x^0, x)\}_{x^0 \in R^3, x \in R^3}$ is regular. This means that for each pair of points x^0, x from R^3 there exists a ray $T(x^0, x)$ which connects these points and this ray is unique.*

Remark 1. The requirement **(A3)** may be formulated in the other terms (see, for example [3]). For this we introduce ray coordinates. Let x^0 be a fixed point in R^3, and x be an arbitrary point in R^3. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be an arbitrary unit vector. Let us consider a ray which goes through x^0 and has the tangent vector at the direction of α. Let x be an arbitrary point of this ray then this point can be given with the Riemann (ray) coordinates $x = f(\zeta, x^0)$, where $\zeta = \alpha\tau$, $(\zeta = (\zeta_1, \zeta_2, \zeta_3))$.

(A3a) *For each pair of points x^0, ζ from R^3 the inequality*

$$\left|\frac{\partial f(\zeta, x^0)}{\partial \zeta}\right| \equiv det\left(\frac{\partial f(\zeta, x^0)}{\partial \zeta}\right) \neq 0$$

holds, and the function $f(\zeta, x^0)$ has the inverse one $\zeta = g(x, x^0)$ for which $g(x^0, x^0) = 0$, and

$$J(x) \equiv \left|\frac{\partial g(x, x^0)}{\partial x}\right| = \left[\left|\frac{\partial f(\zeta, x^0)}{\partial \zeta}\right|_{\zeta = g(x, x^0)}\right]^{-1}, \left|\frac{\partial g(x, x^0)}{\partial x}\right|_{x = x^0} = 1.$$

Here $J(x)$ is Jacobian of the transformation from Cartesian to ray (Riemann's) coordinate system.

Remark 2. The requirement **(A1)** means that the rays inside the ε-neighborhood $U_\varepsilon(x^0)$ are straight lines, and outside $U_\varepsilon(x^0)$ are sufficiently arbitrary smooth curves.

2. Existence and uniqueness theorem

Let v_0, m_0, ε, T be fixed positive numbers, $t_0 = \varepsilon / v_0 < T$. We suppose here that $x \in R^3$, $t \in [0, T]$. This section deals with the existence and the uniqueness of the Cauchy problem (1),(2) in the class of functions having form of the dynamic ray series expansion for $x \in R^3$, $t \in [0, T]$.

Theorem 1. *Let x^0 be a fixed point. Under assumptions **(A1)**–**(A3)** there exists unique generalized solution having the form*

$$(3) \qquad u(x, x^0, t) = \theta_0(t)[\sum_{k=-1}^{N} \sigma_k(x)\theta_k(t - \tau(x, x^0)) + u^N(x, t)],$$

where $\theta_{-1}(\Gamma) = \delta(\Gamma)$ *is the Dirac delta function.*

$$\theta_0(\Gamma) = \begin{cases} 1, \ \Gamma \geq 0, \\ 0, \ \Gamma < 0 \end{cases}$$

is the Heaviside function and

$$\theta_k(\Gamma) = \frac{(\Gamma)^k}{k!}\theta_0(\Gamma), \qquad k = 1, 2, 3, \ldots;$$

the functions $\sigma_k(x)$, $\quad k = -1, 0, 1, 2, 3, \ldots$ *for* $x \in U_\varepsilon(x^0)$, $t \in [0, t_0]$ *are defined by*

$$\text{(4)} \qquad \sigma_{-1}(x) = \frac{1}{4\pi v_0^2 |x - x^0|}, \quad \sigma_k(x) = 0, \qquad k = 0, 1, 2, \ldots$$

$$u^N(x,t) \equiv 0 \quad \textit{for} \quad x \in U_\varepsilon(x^0), \quad t \in [0, t_0];$$

the functions $\sigma_k(x)$, $\quad k = -1, 0, 1, 2, \ldots$ *for* $x \in R^3 \backslash U_\varepsilon(x^0)$, $t \in [t_0, T]$ *are defined by*

$$\text{(5)} \qquad \sigma_{-1}(x(\sigma)) = \frac{1}{4\pi v_0^2 \varepsilon}\sqrt{\frac{v(x)m(x)}{v_0 m_0 J(x)}},$$

$$\sigma_k(x) = \sqrt{\frac{v(x)m(x)}{J(x)}} \times$$

$$\text{(6)} \quad \int_{T(x^0,x)} \frac{1}{2}\sqrt{\frac{v(\xi)J(\xi)}{m(\xi)}}[\nabla_\xi(\ln m(\xi))\nabla_\xi \sigma_{k-1}(\xi) - \Delta_\xi \sigma_{k-1}(\xi)]|d\xi|,$$

$$k = 0, 1, 2, 3, \ldots, N;$$

the function $u^N(x,t)$ *for* $x \in R^3 \backslash U_\varepsilon(x^0)$, $t \in [t_0, T]$ *has the form* $u^N(x,t) = \theta_{N+1}(t - \tau(x, x^0))\tilde{u}(x,t)$, *where* $\tilde{u}(x,t)$ *is a function from* $C^{N-1}(R^3 \times [t_0, T])$.

Proof. Substituting (3) into (1) and using tools of generalized functions theory and reasoning from the works [3, pp.111–121], [4] and the following properties of the Dirac delta and Heaviside functions

$$\delta(t) \cdot \delta(t - \tau(x, x^0)) = 0, \quad \delta'(t) \cdot \theta_k(t - \tau(x, x^0)) = 0, \quad k = 0, 1, 2, \ldots;$$

$$\delta'(t) \cdot \delta(t - \tau(x, x^0)) = -4\pi\tau(x, x^0)v_0^3\delta(x - x^0, t);$$

$$\frac{\partial}{\partial t}\theta_k(\Gamma) = \theta_{k-1}(\Gamma), k = 0, 1, 2, \ldots;$$

$$\frac{\partial}{\partial t}\theta_{-1}(\Gamma) = \delta'(\Gamma), \quad \Gamma = t - \tau(x, x^0);$$

we find the following recurrence relations for $\sigma_k(x)$, $k=-1,0,1,...,N$

$$(7)\qquad 2\nabla\sigma_{-1}(x)\nabla\Gamma-\sigma_{-1}(x)\nabla(\ln m(x))\nabla\Gamma+\sigma_{-1}(x)\Delta\Gamma=0,$$

and

$$2\nabla\sigma_k(x)\nabla\Gamma-\sigma_k(x)\nabla(\ln m(x))\nabla\Gamma+\sigma_k(x)\Delta\Gamma$$

$$(8)\qquad =\nabla(\ln m(x))\nabla\sigma_{k-1}(x)-\Delta\sigma_{k-1}(x),\quad \mathrm{k}=0,1,2,...;$$

and equations for the remainder term $u^N(x,t)$

$$(9)\qquad u_{tt}^N=v^2(x)\Delta u^N-v^2(x)\nabla(\ln m(x))\nabla_x u^N+f_N(x,t),$$

$$(10)\qquad u^N(x,t_0)=0,\qquad u_t^N(x,t_0)=0,$$

where

$$f_N(x,t)=v^2(x)[\Delta\sigma_N(x)-\nabla(\ln m(x))\nabla\sigma_N(x)]\theta_N(\Gamma).$$

Since functions $m(x)$, $v(x)$ are infinitely differentiable then $J(x)$, $\tau(x,x_0)$, $\sigma(x)$ are infinitely differentiable for $|x-x_0|\geq\varepsilon$. Hence $f_N(x,t)\in H^N(R^3\times(t_0,T))$. Applying technique of the work [1, pp.98–105] we obtain formulas (4)–(6) from equations (7),(8). Using hyperbolic equations theory to the problem (9),(10) we can find the existence and uniqueness of the solution of (9),(10) $u^N(x,t)$ having the form

$$u^N(x,t)=\theta_{N+1}(t-\tau(x,x^0))\tilde{u}(x,t),\tilde{u}(x,t)\in C^{N-1}(R^3\times[t_0,T]).$$

3. Formal relation of the dynamic and classical ray series

Let us consider the solution $u(x,t)$ of the problem (1), (2) in the form (3) in which $N=\infty$, $t\in R$, $x\in R^3$, that is

$$(11)\qquad u(x,t)=\theta_0(t)\sum_{k=-1}^{\infty}\sigma_k(x)\theta_k(t-\tau(x,x^0)).$$

A formal application of the Fourier transform with respect to the time variable $t\in R$ to the series (11) leads to classical ray series (see [1])

$$(12)\qquad \tilde{u}(x,\omega)=e^{i\omega\tau(x,x^0)}\sum_{k=-1}^{\infty}\sigma_k(x,x^0)\frac{1}{(-i\omega)^{k+1}},$$

where

$$\tilde{u}(x,t) = F_t[u(x,t)](\omega)$$

$$\equiv F_t[u(x,t)](\omega) = \int_{-\infty}^{\infty} e^{i\omega t} u(x,t)dt, \quad x \in R^3, \quad t \in R.$$

For proving it we need to use the following properties of the Fourier transform.

$$F_t[f](\omega) = \frac{1}{(-i\omega)^m} F_t[\frac{d^m f(t)}{dt^m}](\omega),$$

$$F_t[\theta_k(t - \tau(x,x^0))](\omega) = \frac{1}{(-i\omega)^{k+1}} F_t[\delta(t - \tau(x,x^0))](\omega)$$
$$= \frac{1}{(-i\omega)^{k+1}} e^{i\omega\tau(x,x^0)}, \qquad k = -1, 0, 1, 2, \ldots.$$

Remark 3. Consider the acoustic equation in the frequency domain

$$\omega^2 \tilde{u} + v^2(x)\Delta\tilde{u} - v^2(x)\nabla(\ln m(x))\nabla\tilde{u} + \delta(x - x^0) = 0. \tag{13}$$

If we will seek a solution of this equation in the form of the classical ray series (12) applying the reasoning of the work [1], we get the relations (7), (8) for $\sigma_k(x)$. We note also that in this case we find a formal solution of the equation (13).

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