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# Best approximation and hierarchical bases

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**Summary.** In the present paper we extend the definition of a hierarchical basis to the case of Banach spaces with higher smoothness order elements. Hierarchical bases are similar to the well-known Faber – Schauder system. We prove that there exists a hierarchical basis of the Banach space under consideration. We study the interplay between hierarchical bases and solutions to specific problems of best approximation. The construction of some hierarchical basis is finally described.

**Key words:** hierarchical bases, reflexive Banach spaces, best approximation, reproducing mappings, extremal functions of cubature formulas, splines of affine varieties

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## 1. Introduction

Solutions to most types of problems in applied mathematics are members of a given Banach space and we find it convenient to look for the solution to a problem of such kind in the form of a convergent series with respect to a given basis of the initial Banach space. In particular, in the space $C[0,1]$ of continuous functions with domain $[0,1]$ the well-known Faber — Schauder sequence constitutes a basis (see, e.g., [9, p. 227]) and for each function from $C[0,1]$ the corresponding series is convergent in the norm of $C[0,1]$. It was shown in [4] and [5] that representation of functions in the form of the Faber — Schauder series for problems like interpolation and numerical quadrature has a long tradition. For partial differential equations a similar approach was studied in [1], [4], [5], and [17].

It seems to be several properties of the Faber — Schauder basis which are of crucial importance for problems in the theory of approximation and numerical analysis. In particular, for a given continuous function every coefficient of the corresponding series with respect to the Faber — Schauder system is completely determined by the values of the initial function in the finite subset of its domain. But we can not use the Faber — Schauder system as a basis in case when the functions of a given Banach space has derivatives of order greater than 1. In this event it is natural to ask

Does there exist a basis like the Faber — Schauder system for a given Banach space with functional members of high order of smoothness?

Throughout the sequel we call bases like the Faber — Schauder system the hierarchical bases.

In general the hierarchical basis can inherit the properties of the Faber — Schauder system only in part. An example is as follows.

Let $\Omega \subset R^n$ be a bounded domain and let $B$ be a Banach space of harmonic functions with domain $\Omega$. If $u$ is a member of some basis of $B$ then the support of $u$ can not be in the interior of $\Omega$. For, were it otherwise, we would find that $u$ is identically equal to 0; a contradiction.

Nevertheless, we can define a hierarchical basis in a separable Banach space of rather general type and the hierarchical basis defined in such a way inherits some important properties of the Faber — Schauder system. The goal of the present paper is to describe the construction of some hierarchical basis in a separable reflexive Banach space. By this way we also establish the existence of a hierarchical basis in the space under consideration.

Let $\Omega \subset R^n$ be a bounded domain with sufficiently smooth boundary and let the origin be in the interior of $\Omega$.

The setting is a separable reflexive Banach space $X = X(\Omega)$ and a reflexive Banach space $Y = X^*$, dual to $X$. The members of $X$ are real valued continuous functions with domain $\overline{\Omega}$. Let $X$ be embedded in the Banach space $C(\overline{\Omega})$ of functions which are continuous in $\overline{\Omega}$ and let the embedding be linear and bounded. Hence the conventional Dirac delta function $\delta(x)$ is a member of $Y$.

We also assume that for a given finite subset $F$ of $\overline{\Omega}$ there exists a member $u(x)$ of $X$ such that the values of $u(x)$ at points of $F$

are prescribed real numbers. It will be true if, for example, every polynomial belongs to $X$.

Furthermore, let $X$ be a *strictly normed* linear space, i.e.,

$$\|u + v \mid X\| = \|u \mid X\| + \|v \mid X\|$$

implies that $u = tv$ for some $t \geq 0$ or else $v = 0$. This constraint on the norm is easily seen to be equivalent to the geometric condition that the unit ball of $X$ be rotund. Since $X$ is strictly normed it follows that $Y = X^*$ is a *smoothly normed* reflexive Banach space [8, p. 173]. Let $N^*(\cdot) = \|\cdot \mid Y\|$. By a definition, $Y$ is smoothly normed exactly when

$$(N_G^{*,\prime}(l), m) = \lim_{t \to 0} \frac{1}{t}(N^*(l + tm) - N^*(l))$$

exists for all $m, l \in Y$, $\|l \mid Y\| = 1$, and defines a functional $N_G^{*,\prime}(l)$ in $Y^* = X$. The functional $N_G^{*,\prime}(l)$ is a Gateaux differential of the norm $N^*(\cdot)$ at $l$.

Let $Y$ be also a *strictly normed* linear space. Then there exists a Gateaux differential of the norm $N(\cdot) = \|\cdot \mid X\|$ at all unit vectors $u \in X$.

Examples of strictly and smoothly normed spaces are Hilbert spaces and the Sobolev spaces $W_p^{(m)}(\Omega)$; $1 < p < \infty$. By a definition, $\varphi(x)$ with domain $\Omega$ belongs to $W_p^{(m)}(\Omega)$ iff $\varphi$ have all derivatives up to order $m$ locally integrable and

$$\|\varphi \mid W_p^{(m)}(\Omega)\|^p = \int\limits_\Omega \{|\varphi|^2 + \sum_{|\alpha| = m} \frac{m!}{\alpha!}|D^\alpha \varphi|^2\}^{p/2}\, dx < \infty.$$

The integral here spreads over $\Omega$, and summation is taken over some multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with integer coefficients,

$$\alpha! = \alpha_1!\alpha_2!\ldots\alpha_n!, \quad |\alpha| = \sum_{j=1}^n \alpha_j, \quad D^\alpha \varphi = \frac{\partial^m \varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

For $mp > n$ $W_p^{(m)}(\Omega)$ is embedded in $C(\Omega)$; and the embedding is linear and bounded.

Let $\boldsymbol{\Delta} = \{\Delta_k\}_{k=0}^\infty$ be a sequence of finite subsets of $\overline{\Omega}$; $\Delta_0 = \{\tilde{x}_j^{(0)} \mid j = 1, 2, \ldots, \sigma(0)\}$, $\Delta_k = \Delta_{k-1} \cup \{\tilde{x}_j^{(k)} \mid j = 1, 2, \ldots, \sigma(k)\}$, $\tilde{x}_j^{(k)} \notin \Delta_{k-1}$, $k = 1, 2, \ldots$. We assume that the union of all $\Delta_k$ is dense in $\overline{\Omega}$. The sequence $\boldsymbol{\Delta} = \{\Delta_k\}_{k=0}^\infty$ is said to be *a multigrid* in $\overline{\Omega}$; $\Delta_k$ is *a $k$-level* of $\boldsymbol{\Delta}$; and vectors $\tilde{x}_j^{(k)} \in \overline{\Omega}$ are *nodes* of $\boldsymbol{\Delta}$.

Given a multigrid $\boldsymbol{\Delta}$, we introduce the sequence $\mathrm{Nod}_j^{(k)}$ of subsets of $\boldsymbol{\Delta}$ by putting $\mathrm{Nod}_1^{(0)} = \emptyset$ and for $k = 0, 1, 2, \dots$

$$\begin{aligned} \mathrm{Nod}_j^{(k)} &= \mathrm{Nod}_{j-1}^{(k)} \cup \{\tilde{x}_{j-1}^{(k)}\}, \quad j = 2, 3, \dots, \sigma(k); \\ \mathrm{Nod}_{\sigma(k)+1}^{(k)} &= \mathrm{Nod}_1^{(k+1)} = \Delta_k. \end{aligned}$$

If $k < k_1$ or ($k = k_1$ and $j \le j_1$), then

$$\Delta_{k-1} \subset \mathrm{Nod}_j^{(k)} \subset \mathrm{Nod}_{j_1}^{(k_1)} \subset \Delta_{k_1}.$$

Let $H = \{h_j^{(k)} \in X \mid k \ge 0,\ j = 1, \dots, \sigma(k)\}$ be a countable subset of $X$.

**Definition 1.** *We call $H$ a $\boldsymbol{\Delta}$-hierarchical system in $X$ iff for all $k$, $j$, and $\tilde{x}_i^{(m)} \in \Delta_k$, $\tilde{x}_i^{(m)} \ne \tilde{x}_j^{(k)}$, the function $h_j^{(k)}$ equals 0 at $\tilde{x}_i^{(m)}$; and the value of $h_j^{(k)}$ at $\tilde{x}_j^{(k)}$ equals* 1.

Hence, $H$ is a $\boldsymbol{\Delta}$–hierarchical system in $X$ iff the following equalities hold

$$(1.1) \qquad h_j^{(k)}(\tilde{x}_l^{(m)}) = \delta_j^l \delta_k^m; \quad m = 0, 1, \dots, k, \quad l = 1, 2, \dots, \sigma(k).$$

Here $\delta_j^l$ is the conventional Kronecker delta.

Given the finite subset $\Delta_k$ of $\Omega$ and an integer $j, j = 1, 2, \dots, \sigma(k)$, we can apply the Lagrange interpolation formula over the set $\Delta_k$ of nodes and find $h_j^{(k)} \in X$ such that (1.1) holds. Hence the set of $\boldsymbol{\Delta}$–hierarchical systems of $X$ is not empty. Obviously, every finite subsequence of a $\boldsymbol{\Delta}$–hierarchical system $H$ is linearly independent.

In [12] and [2] it was shown that hierarchical systems in spaces like Sobolev spaces may be constructed as sequences of interpolating $D^m$–splines. The properties of $\boldsymbol{\Delta}$–hierarchical systems in Hilbert spaces were studied in [3]. Cubature formulas based on hierarchical systems were constructed in [15] and [16].

**Definition 2.** *If a $\boldsymbol{\Delta}$-hierarchical system $H$ in $X$ is a basis of $X$, then $H$ is called a hierarchical basis.*

An example of a hierarchical basis for the space $C[0,1]$ and the Sobolev space $W_2^1[0,1]$ simultaneously is the well-known Faber — Schauder system.

We find it convenient to formulate a general problem as follows.

**Problem 1.** *Given a separable Banach space $X$, find a $\boldsymbol{\Delta}$-hierarchical basis of $X$.*

Alongside Problem 1, it stands to reason to consider another problem that is posed in the theory of approximation. To be more precise, we bear in mind the problem of best approximation in a Banach space. Before stating the problem, we introduce a few designations.

Since $\mathrm{Nod}_j^{(k)} \subset \overline{\Omega}$ it follows that for all $\tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}$ the functional $\delta(x - \tilde{x}_i^{(m)})$ is a member of $Y$. Evidently the linear span

$$L_j^{(k)} = \mathrm{span}\{\delta(x - \tilde{x}_i^{(m)}) \mid \tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}\}$$

is a finite dimensional closed subspace of $Y$.

**Problem 2.** *Given an arbitrary nonzero functional $l_1^{(0)} \in Y$, find an element of best approximation to $l_1^{(0)}$ from $L_j^{(k)}$.*

As is known, every closed convex subset of a reflexive strictly normed Banach space is a Chebyshev set (see, e.g.,[10, p. 104]). It means that every element of the Banach space has exactly one element of best approximation from the set under consideration. Consequently, there exists a unique solution to Problem 2.

The theme of our presentation up to this point may be described as a study of the interplay between the solutions of Problems 1 and 2.

To begin with, we take $l_1^{(0)} \in Y$, $l_1^{(0)} \neq 0$. For example, $l_1^{(0)}$ may be the indicator $\chi_\Omega(x)$ of $\Omega$. To $l_1^{(0)}$ and every vector $\{c_i^{(m)} \mid \tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}\}$ of real numbers, we assign the associated sequence of error functionals by putting

$$l_j^{(k)} = l_1^{(0)} - \sum_{\tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}} c_i^{(m)} \delta(x - \tilde{x}_i^{(m)}),\ k = 0, 1, \ldots,\ j = 1, 2, \ldots, \sigma(k).$$

We call the sequence $\{l_j^{(k)}\}$ the *error multifunctional*. The corresponding sequence of cubature formulas is said to be *a multicubature formula.*

Let $Y_j^{(k)}$ be a flat parallel to $L_j^{(k)}$; $Y_j^{(k)} = l_1^{(0)} + L_j^{(k)}$. Then $Y_j^{(k)}$ is an affine variety and

$$\dim Y_j^{(k)} = N = N(k, j) = \sigma(0) + \sigma(1) + \cdots + \sigma(k-1) + j - 1.$$

If $k < k_1$ or ($k = k_1$ and $j \leq j_1$), then $Y_j^{(k)} \subset Y_{j_1}^{(k_1)}$.

For given integers $k$ and $j$, we use the symbol $\delta^{(k)}_{j,\mathrm{opt}}(x)$ to designate the element of best approximation to $l^{(0)}_1$ from $L^{(k)}_j$, and coefficients of the expansion of $\delta^{(k)}_{j,\mathrm{opt}}(x)$ with respect to delta functions $\delta(x-\tilde{x}^{(m)}_i)$ we denote by $c^{(m)}_{i,\mathrm{opt}} = c^{(m)}_{i,\mathrm{opt}}(j,k)$, i.e.,

$$\delta^{(k)}_{j,\mathrm{opt}}(x) = \sum_{m=0}^{k-1}\sum_{i=1}^{\sigma(m)} c^{(m)}_{i,\mathrm{opt}}\delta(x-\tilde{x}^{(m)}_i) + \sum_{i=1}^{j-1} c^{(k)}_{i,\mathrm{opt}}\delta(x-\tilde{x}^{(k)}_i).$$

Let $l^{(k)}_{j,\mathrm{opt}} = l^{(0)}_1 - \delta^{(k)}_{j,\mathrm{opt}}$. The corresponding cubature formula is said to be $X$-*optimal* on the set $\mathrm{Nod}^{(k)}_j$ of nodes [14]. The norm $\|l^{(k)}_{j,\mathrm{opt}} \mid Y\|$ equals $E(l^{(0)}_1, L^{(k)}_j)$, where $E(w,N)$ is the distance from $w \in Y$ to a linear subspace $N$ of $Y$.

Let $l \in Y$ and $u \in X$. If the following equalities hold

$$\|l \mid Y\|^2 = (l,u) = \|u \mid X\|^2, \tag{1.2}$$

then $u$ is said to be *an extremal function for* $l$ [14] and $l$ is said to be *a generated extremal function for* $u$.

By the reflexivity of $X$ and James Theorem (see, e.g., [11, p. 236]), there exists an extremal function for an arbitrary $l \in Y$. Since $X$ is a strictly normed space it follows that for a given functional $l \in Y$ an extremal function $u \in X$ is unique. By the same reasons, for a given function $u \in X$ there exists a unique generated extremal function. Throughout the sequel we denote the extremal function for $l^{(k)}_{j,\mathrm{opt}}$ by $u^{(k)}_{j,\mathrm{opt}}$.

In Section 5 we discuss how to transform the set

$$\{u^{(k)}_{j,\mathrm{opt}}(x), u^{(k)}_{j+1,\mathrm{opt}}(x), \ldots, u^{(k)}_{\sigma(k),\mathrm{opt}}(x)\}$$

into the function $h^{(k)}_j(x)$ of some $\boldsymbol{\Delta}$-hierarchical basis of $X$; and it is the main result of the paper.

## 2. Extremal Functions and Reproducing Mappings

Let $l^{(0)}_1 \in Y$, $l^{(0)}_1 \neq 0$, and let $u^{(0)}_1$ be the extremal function for $l^{(0)}_1$. Then

$$\|l^{(0)}_1 \mid Y\|^2 = (l^{(0)}_1, u^{(0)}_1) = \|u^{(0)}_1 \mid X\|^2. \tag{2.1}$$

As our next step, we consider the properties of $u^{(0)}_1$.

**Theorem 1.** *There is a unique element $u = u(x)$ of best approximation to zero element of $X$ from*

$$V = \{v \in X \mid (l_1^{(0)}, v) = \|l_1^{(0)}\|^2\}.$$

*The function $u$ is a unique extremal function for $l_1^{(0)}$ in $X$. If $M$ is the kernel of $l_1^{(0)}$ and $E(w, N)$ is the distance from $w \in X$ to a linear subspace $N$ of $X$, then*

$$E(u, M) = E(0, V) = \|l_1^{(0)} \mid Y\| = \|u \mid X\|.$$

*Proof.* Let $v_* \in X$, $(l_1^{(0)}, v_*) = d \neq 0$, and $\alpha = \|l_1^{(0)}\|^2/d$. Then $v_0 = \alpha v_* \in V$. Consequently, $V = v_0 + M \neq \emptyset$, and $V$ is a closed convex subset of $X$. Whence and from the reflexivity of $X$, we infer that $V$ is a Chebyshev subset of $X$, and there exists a unique element $u = u(x)$ of best approximation to zero element of $X$ from $V$. By the definition,

$$\|u\| = \min\{\|v\| \mid v \in V\} = \min\{\|u - v\| \mid v \in M\},$$

and zero element of $X$ is an element of best approximation to $u$ from $M$. Whence and from the well-known theorem of characterization of elements of best approximation [13, p. 2] it follows that there exists $f_0 \in Y$ such that

$$\|f_0\| = 1; \quad \|u\| = (f_0, u); \quad (f_0, v) = 0 \ \forall v \in M. \tag{2.2}$$

If $v_* \in X$ and $(l_1^{(0)}, v_*) = d \neq 0$, then for $\forall\, w \in X$ we have $w = \alpha v_* + v$, where $\alpha = (l_1^{(0)}, w)/d$ and $v \in M$. By the third equality of (2.2), $(f_0, w) = \alpha(f_0, u_*) = \beta(l_1^{(0)}, w)$, where $\beta = (f_0, u_*)/d$. Therefore, $f_0 = \beta l_1^{(0)}$ and $(f_0, u) = \beta(l_1^{(0)}, u) = \beta\|l_1^{(0)}\|^2$. By the second equality of (2.2), $\beta = \|u\|/\|l_1^{(0)}\|^2$. Considering this, we derive $\beta > 0$, and, by the first equality of (2.2), $\beta = 1/\|l_1^{(0)}\|$. Hence $\|u\| = \|l_1^{(0)}\|$, and we arrive at the sought relations (2.1) for the function $u_1^{(0)} = u$.

From the strict convexity of $X$ it is immediate that there is a unique extremal function for $l_1^{(0)}$ in $X$. For, were it otherwise, we would find at least two extremal functions $u_1$ and $u_2$ with the same norm and their half-sum $u_{12} = (u_1 + u_2)/2$ would then have the norm less than each of them. In this event,

$$\|l_1^{(0)}\| \geq \frac{(l_1^{(0)}, u_{12})}{\|u_{12}\|} > \|l_1^{(0)}\|;$$

a contradiction. $\square$

Applying Theorem 1 to the space $Y$, we obtain

**Theorem 2.** *There is a unique element $l$ of best approximation to zero element of $Y$ from*

$$V^* = \{m \in Y \mid (m, u_1^{(0)}) = \|u_1^{(0)}\|^2\}.$$

*The functional $l$ is a unique generated extremal function for $u_1^{(0)}$ in $Y$. If $M^* = \{m \in Y \mid (m, u_1^{(0)}) = 0\}$ and $E(l, N^*)$ is the distance from $l \in Y$ to a linear subspace $N^*$ of $Y$, then*

$$E(l, M^*) = E(0, V^*) = \|u_1^{(0)} \mid X\| = \|l \mid Y\|.$$

Let $l_1^{(0)} \in Y$, $u_1^{(0)} \in X$, $u_1^{(0)} \neq 0$, and (2.1) holds. Then we can define the mapping $\pi : Y \to X$ by $\pi(l_1^{(0)}) = u_1^{(0)}$. We also assume that $\pi(0) = 0$. By the definition, $\pi(l_1^{(0)})$ is the extremal function for $l_1^{(0)} \in Y$. By Theorem 1, it follows that $\pi$ is a single-valued mapping with domain $Y$ and for $l \in Y$ and $\alpha \in R$ we have

$$\|\pi(l) \mid X\|^2 = (l, \pi(l)) = \|l \mid Y\|^2, \quad \pi(\alpha l) = \alpha\pi(l).$$

Together with $\pi$, we consider a mapping $\pi^* : X \to Y$, dual to $\pi$. Let $u_1^{(0)} \in X$, $u_1^{(0)} \neq 0$, $l_1^{(0)} \in Y$, and (2.1) holds. We assume that $\pi^*(u_1^{(0)}) = l_1^{(0)}$ and $\pi^*(0) = 0$. By Theorem 2, it follows that $\pi^*$ is a single-valued mapping with domain $X$ and

$$\|\pi^*(u) \mid Y\|^2 = (\pi^*(u), u) = \|u \mid X\|^2, \quad \pi^*(\alpha u) = \alpha\pi^*(u), \ \forall\, u \in X.$$

Let $l \in Y$ and $u \in X$. By Theorems 1 and 2 $l = \pi^*(u)$ iff $u = \pi(l)$. In particular for $l \in Y$ and $u \in X$

$$l = \pi^*(\pi(l)) \quad \text{and} \quad u = \pi(\pi^*(u)).$$

Hence $\pi : Y \to X$ and $\pi^* : X \to Y$ are reciprocal and surjective mappings. The image of the sphere of radius $R$ under the mapping $\pi$ is the sphere of the same radius $R$. Conversely, the image of the sphere of radius $R$ under the mapping $\pi^*$ is the sphere of the same radius $R$.

If $X$ is a Hilbert space, then $\pi = \pi^*$ and $\pi$ is said to be *reproducing mapping of $X$* [2, p. 23]. We also find it convenient to use the term "reproducing mapping" in the case of a Banach space. To be more precise, we call $\pi$ (resp. $\pi^*$) the reproducing mapping of the Banach space $Y$ (resp. $X$). In [8, p. 174] $\pi^*$ is called *the norm-duality map.*

Because of developments in the abstract theory of convex programming, it is possible to readily characterize the extremal function for an arbitrary functional $l_1^{(0)} \in Y$, $l_1^{(0)} \neq 0$. By hypothesis $X$ is a smoothly normed space. In this case let

$$(2.3) \qquad (N'_G(v), w) = \lim_{t \to 0} \frac{1}{t}(N(v + tw) - N(v));$$

this is defined (by assumption) whenever $v \neq 0$ and $N'_G(v)$ is a Gateaux differential of the norm $N(\cdot) = \|\cdot \mid X\|$ at $v$; $N'_G(v) \in Y$. We have

**Theorem 3.** *Let $l_1^{(0)}$ be a nonzero element of $Y$; $M$ is the kernel of $l_1^{(0)}$; $u \in X$ and $(l_1^{(0)}, u) = \|l_1^{(0)}\|^2$. The function $u$ is extremal for $l_1^{(0)}$ iff $(N'_G(u), w) = 0$ for $\forall\, w \in M$. Moreover, the extremal function $u \in X$ for $l_1^{(0)}$ is the solution to the following problem*

$$(2.4) \qquad \begin{cases} N'_G(u) \; = \dfrac{1}{\|l_1^{(0)}\|} l_1^{(0)}, \\ \\ (l_1^{(0)}, u) = \|l_1^{(0)}\|^2. \end{cases}$$

*Conversely, every solution $u \in X$ to* (2.4) *is the extremal function for $l_1^{(0)}$. There is a unique solution to* (2.4).

*Proof.* Let $V = \{v \in X \mid (l_1^{(0)}, v) = \|l_1^{(0)}\|^2\} = v_0 + M$ and $u \in V$. By Theorem 1, $u$ is the extremal function for $l_1^{(0)}$ iff

$$\|u\| = \min\{\|v\| \mid v \in V\}.$$

Consequently, $u$ is an $R$-spline interpolant of $V$, with $R$ the identity map on $X$ [6, p. 576]. Whence and from Corollary 3.1 [6, p. 584] it follows that $u \in V$ is the extremal function for $l_1^{(0)}$ iff $(N'_G(u), w) = 0$ for $\forall\, w \in M$.

Let $u$ be the extremal function for $l_1^{(0)}$, $u \in X$. If $\varphi \in X$ then $\varphi = \alpha u + w$, where $w \in M$ and $\alpha \in R$. Considering this, we derive

$$(N'_G(u), \varphi) = \alpha(N'_G(u), u) + (N'_G(u), w).$$

Since (2.3) holds, we have $(N'_G(u), u) = \lim_{t \to 0} \frac{1}{t}(\|u + tu\| - \|u\|) = \|u\|$. But $(N'_G(u), w) = 0$ for $\forall\, w \in M$, and we obtain $\alpha = (N'_G(u), \varphi)/\|u\|$. Inserting this equality in $(l_1^{(0)}, \varphi) = \alpha(l_1^{(0)}, u)$ and considering that $u$ is the extremal function for $l_1^{(0)}$, we find that

$$\frac{1}{\|l_1^{(0)}\|}(l_1^{(0)}, \varphi) = \frac{(l_1^{(0)}, u)}{\|u\|^2}(N'_G(u), \varphi) = (N'_G(u), \varphi).$$

Hence, $u$ is actually a solution to (2.4).

Assume now that $u$ is a solution to (2.4). Then $(N'_G(u), w) = \frac{1}{\|l_1^{(0)}\|}(l_1^{(0)}, w) = 0$ for $\forall\, w \in M$. In this event, as we know, the function $u$ is extremal for $l_1^{(0)}$.

By Theorem 1 there is a unique extremal function for $l_1^{(0)}$ in $X$. Hence, there is a unique solution of (2.4). □

By hypotheses of Theorem 3, the image of $l_1^{(0)} \in Y$ under the mapping $\pi$ is the unique solution to (2.4).

**Lemma 1.** *Let $X$ be a Hilbert space with the inner product $(\cdot,\cdot)_X$ and let $l_1^{(0)}$ be a nonzero member of $Y$. The extremal function $u_1^{(0)}$ for $l_1^{(0)}$ satisfies the following equalities*

$$(l_1^{(0)}, \varphi) = (u_1^{(0)}, \varphi)_X, \quad \forall \varphi \in X. \tag{2.5}$$

*Thus, $u_1^{(0)}$ is the member of $X$ associated to the given functional by virtue of the Riesz Theorem on the general form of a bounded linear functional.*

*Proof.* Let $X$ be a Hilbert space. In this event the Gateaux differential $N'_G(v)$ of the norm $N(\cdot)$ at $u_1^{(0)}$ is defined by

$$(N'_G(u_1^{(0)}), w) = \frac{1}{\|u_1^{(0)}\|}(u_1^{(0)}, w)_X, \quad \forall w \in X.$$

Whence and from Theorem 3 it follows that (2.5) holds. □

The following theorem is dual to Theorem 3.

**Theorem 4.** *Let $u_1^{(0)}$ be a nonzero element of $X$;*

$$M^* = \{m \in Y \mid (m, u_1^{(0)}) = 0\}$$

*be the annihilator of $\{u_1^{(0)}\} \subset X$; $l \in Y$, and $(l, u_1^{(0)}) = \|u_1^{(0)}\|^2$. Then $l$ is a generated extremal function for $u_1^{(0)}$ iff $(m, N_G^{*,\prime}(l)) = 0$ for $\forall\, m \in M^*$. Moreover, the generated extremal function $l \in Y$ for $u_1^{(0)}$ is the solution to the following problem*

$$\begin{cases} N_G^{*,\prime}(l) \; = \dfrac{1}{\|u_1^{(0)}\|} u_1^{(0)}, \\ \\ (l, u_1^{(0)}) = \|u_1^{(0)}\|^2. \end{cases} \tag{2.6}$$

*Conversely, every solution $l \in Y$ to (2.6) is the generated extremal function for $u_1^{(0)}$. There is a unique solution to (2.6).*

By hypotheses of Theorem 4, the image of $u_1^{(0)} \in Y$ under the mapping $\pi^*$ is the unique solution to (2.6).

**Theorem 5.** *The reproducing mappings $\pi$ and $\pi^*$ are demicontinuous and the following inequalities hold*

$$(\pi^*(u) - \pi^*(v), u - v) \geq 0, \quad \forall\, u, v \in X;$$

$$(\pi(l) - \pi(m), l - m) \geq 0, \quad \forall\, l, m \in Y,$$

*i.e., $\pi$ and $\pi^*$ are monotone. If for each nonzero $v \in X$ (resp. $l \in Y$) the functional $N_G'(v)$ (resp. $N_G^{*,\prime}(l)$) is the Frechet differential of the norm at $v$ (resp. $l$), then $\pi^*$ (resp. $\pi$) is continuous.*

*Proof.* To begin with, we consider the reproducing mapping $\pi^*$. Let $u \in X$, $u \neq 0$; and $l = \pi^*(u)$. By (2.4), the following equalities hold

$$l = \|l\| N_G'(u) = \|u\| N_G'(u) = N_G'(\|u\| u).$$

In terms of [8, p. 174] $\pi^*$ is the norm-duality map from $X$ into $Y$. There are proofs of the monotonicity inequality for $\pi^*$ and the demicontinuity of $\pi^*$ in [8, p. 174].

Let for each nonzero $v \in X$ the functional $N_G'(v)$ be the Frechet differential of $\|\cdot \mid X\|$ at $v$. Under this hypothesis, there is a proof of the continuity of $\pi^*$ in [7, p. 149].

By the same way, we establish the properties of $\pi$. $\square$

If there is a Frechet differential of $\|\cdot \mid X\|$ at $v \in X$, $v \neq 0$, and a Frechet differential of $\|\cdot \mid Y\|$ at $l \in Y$, $l \neq 0$, then it follows from Theorem 5 that $\pi^*$ and $\pi$ are homeomorphisms of $X$ and $Y$.

Since the affine variety $Y_j^{(k)}$ of error functionals is an unbounded subset of $Y$ it follows that the image $X_j^{(k)}$ of $Y_j^{(k)}$ under $\pi$ is an unbounded subset of $X$. If $k < k_1$ or ($k = k_1$ and $j \leq j_1$), then $X_j^{(k)} \subset X_{j_1}^{(k_1)}$. Let $\pi^*$ be continuous. Since $Y_j^{(k)}$ is a closed subset of $Y$ it follows that $X_j^{(k)}$ is a closed subset of $X$.

There is a one-to-one correspondence $\tau_j^{(k)}$ of topological space $X_j^{(k)}$ onto $R^{N(j,k)}$, where $N(j,k) = \dim Y_j^{(k)}$. The definition of $\tau_j^{(k)}$ is as follows.

Let $H$ be a hierarchical system in $X$, $h_i^{(m)} \in H$, $\tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}$, and $u \in X_j^{(k)}$. Then we assume that

$$c_i^{(m)}(u) = (l_1^{(0)} - \pi^*(u), h_i^{(m)}).$$

Let $\tau_j^{(k)}(u) = \{c_i^{(m)} \mid \tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}\} \in R^{N(j,k)}$. Examine that $\tau_j^{(k)}$ is actually a one-to-one correspondence of $X_j^{(k)}$ onto $R^{N(j,k)}$.

Let $u_1 \in X_j^{(k)}$, $u_2 \in X_j^{(k)}$, and $\tau_j^{(k)}(u_1) = \tau_j^{(k)}(u_2)$. Then $l_1 = \pi^*(u_1) \in Y_j^{(k)}$, $l_2 = \pi^*(u_2) \in Y_j^{(k)}$, and for $\forall\, \tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}$ we have $(l_1 - l_2, h_i^{(m)}) = 0$. Considering that $l_1 - l_2 \in L_j^{(k)}$, we arrive at $l_1 = l_2$. Hence $u_1 = \pi(l_1) = \pi(l_2) = u_2$.

If $\pi^*$ is continuous then $\tau_j^{(k)}$ is also continuous. In this event $X_j^{(k)}$ is a topological variety of dimension $N(j,k) = \dim Y_j^{(k)}$ and $X_j^{(k)}$ is homeomorphic to the affine variety $Y_j^{(k)}$.

In case of a Hilbert space the extremal function for $l_j^{(k)} \in Y_j^{(k)}$ is given by the formula $u_j^{(k)}(x) = u_1^{(0)}(x) - \sum\limits_{\tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}} c_i^{(m)} U_{\delta,i}^{(m)}(x)$, where $U_{\delta,i}^{(m)}(x)$ is the extremal function for $\delta(x - \tilde{x}_i^{(m)})$. If $U_\delta(x)$ is the extremal function for the Dirac delta function $\delta(x)$ and $U_\delta(x - \tilde{x}_i^{(m)}) \in X$, then $U_{\delta,i}^{(m)}(x) = U_\delta(x - \tilde{x}_i^{(m)})$.

**Lemma 2.** *The norm of the extremal function $u_{j,\mathrm{opt}}^{(k)}$ for the optimal error functional $l_{j,\mathrm{opt}}^{(k)}$ is less than the norm of an arbitrary element of $X_j^{(k)}$;*

$$u_{j,\mathrm{opt}}^{(k)} = \arg\min\{\|v \mid X\| \mid v \in X_j^{(k)}\}.$$

*There is a unique element of $X_j^{(k)}$ with this property.*

*Proof.* Let $u \in X_j^{(k)}$; $u \neq u_{j,\mathrm{opt}}^{(k)}$. Then $l = \pi^*(u) \in Y_j^{(k)}$ and $l \neq l_{j,\mathrm{opt}}^{(k)}$. Hence $\|u \mid X\| = \|l \mid Y\| > \|l_{j,\mathrm{opt}}^{(k)} \mid Y\| = \|u_{j,\mathrm{opt}}^{(k)} \mid X\|$. $\square$

Let us establish the additional properties of $u_{j,\mathrm{opt}}^{(k)}$. To this end, we apply the following

**Theorem 6.** [10, p. 116] *Let $Y_1$ be a closed linear subspace of $Y$, $l_1^{(0)} \in Y \setminus Y_1$, and $\delta_{\mathrm{opt}} \in Y_1$. The functional $\delta_{\mathrm{opt}}$ is the element of best approximation to $l_1^{(0)}$ from $Y_1$ iff there exists a member $f_0$ of $Y^*$ such that*

$$\|f_0 \mid Y^*\| = 1; \quad \|l_1^{(0)} - \delta_{\mathrm{opt}} \mid Y\| = f_0(l_1^{(0)}); \quad f_0(\delta_c) = 0\ \forall \delta_c \in Y_1.$$

Demonstrate that the following claim is true.

**Theorem 7.** *A function $u \in X_j^{(k)}$ is extremal for optimal error functional $l_{j,\mathrm{opt}}^{(k)}$ iff*

$$u(\tilde{x}_s^{(p)}) = 0 \quad \forall\, \tilde{x}_s^{(p)} \in \mathrm{Nod}_j^{(k)}. \tag{2.7}$$

*Proof.* By the reflexivity of $X$, it follows that for $\forall\, f \in Y^*$ there is a function $u \in X$ such that $f(l) = l(u)$ for $\forall\, l \in Y$. Applying Theorem 6 to the space $Y$ and the subspace

$$Y_1 = L_j^{(k)} = \mathrm{span}\{\delta(x - \tilde{x}_i^{(m)}) \mid \tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}\}$$

and considering that the element of best approximation to $l_1^{(0)}$ from $L_j^{(k)}$ is denoted by $\delta_{j,\mathrm{opt}}^{(k)}$, we conclude that there is a function $u_0 \in X$ such that

$$\|u_0\| = 1;\ \|l_1^{(0)} - \delta_{j,\mathrm{opt}}^{(k)} \mid Y\| = (l_1^{(0)}, u_0);\ \ (\delta_c, u_0) = 0\,\forall \delta_c \in L_j^{(k)}. \tag{2.8}$$

The third condition of (2.8) holds iff

$$u_0(\tilde{x}_s^{(p)}) = 0 \quad \forall\, \tilde{x}_s^{(p)} \in \mathrm{Nod}_j^{(k)}. \tag{2.9}$$

Considering this, we write the second condition of (2.8) in the equivalent form

$$\|l_{j,\mathrm{opt}}^{(k)} \mid Y\| = \|l_1^{(0)} - \delta_{j,\mathrm{opt}}^{(k)} \mid Y\| = (l_1^{(0)} - \delta_{j,\mathrm{opt}}^{(k)}, u_0).$$

If $v(x) = \|l_{j,\mathrm{opt}}^{(k)} \mid Y\| \cdot u_0(x)$, then it follows from the last equality that $\|l_{j,\mathrm{opt}}^{(k)} \mid Y\|^2 = (l_{j,\mathrm{opt}}^{(k)}, v) = \|v \mid X\|^2$. Thus, $v$ is the extremal function for $l_{j,\mathrm{opt}}^{(k)}$, and $v = u_{j,\mathrm{opt}}^{(k)}$. Whence and from (2.9) we infer that (2.7) holds.

Assume now that $u \in X_j^{(k)}$, (2.7) holds, and $c_i^{(m)} = c_i^{(m)}(u)$ are local coordinates of $u \in X_j^{(k)}$, i.e., $c_i^{(m)} = c_i^{(m)}(u)$ are entries of the vector $\tau_j^{(k)}(u) \in R^{N(j,k)}$. Let $\delta_* = \sum_{\tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}} c_i^{(m)} \delta(x - \tilde{x}_i^{(m)})$. Then $\delta_* \in L_j^{(k)}$, and $\tau_j^{(k)}(\pi(l_1^{(0)} - \delta_*)) = \tau_j^{(k)}(u)$. Hence the function $u = \pi(l_1^{(0)} - \delta_*)$ is extremal for $l_1^{(0)} - \delta_*$, and

$$(l_1^{(0)} - \delta_*, u) = \|l_1^{(0)} - \delta_* \mid Y\|^2 = \|u \mid X\|^2. \tag{2.10}$$

The function $u_0(x) = \frac{1}{\|u|X\|} \cdot u(x)$ belongs to the unit sphere of $X$ and, by (2.7), satisfies $(\delta_c, u_0) = 0$ for $\forall \delta_c \in L_j^{(k)}$. Whence and from (2.10) it follows that

$$(l_1^{(0)}, u_0) = (l_1^{(0)} - \delta_*, u_0) = \frac{1}{\|u \mid X\|}(l_1^{(0)} - \delta_*, u)$$

$$= \frac{1}{\|u \mid X\|} \|l_1^{(0)} - \delta_* \mid Y\|^2 = \|l_1^{(0)} - \delta_* \mid Y\|.$$

Thus for given $\delta_* \in L_j^{(k)}$ there is a member $u_0$ of $X = Y^*$ such that (2.8) holds. By Theorem 6, $\delta_*$ is the element of best approximation to $l_1^{(0)}$ from $L_j^{(k)}$, and $l_1^{(0)} - \delta_* = l_{j,\mathrm{opt}}^{(k)}$. Consequently, $u = \pi(l_{j,\mathrm{opt}}^{(k)}) = u_{j,\mathrm{opt}}^{(k)}$. □

As is well known, for a given set of nodes there is a unique optimal cubature formula in a Hilbert space and the corresponding optimal extremal function equals 0 at all nodes of this formula. Moreover, in case of Hilbert spaces like Sobolev spaces, the system (2.7) is a starting point of the algorithm of constructing the (unknown) weights of an optimal cubature formula (see, e.g., [14, Chapter 9]).

Let

$$M_j^{(k)} = \{v \in X \mid (l_1^{(0)}, v) = 0,\ (\delta(x - \tilde{x}_i^{(m)}), v) = 0 \,\forall\, \tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}\}$$

be a closed linear subspace of $X$ and let $V_j^{(k)} = u_{j,\mathrm{opt}}^{(k)} + M_j^{(k)}$ be the flat parallel to $M_j^{(k)}$.

**Theorem 8.** *Let the norm $N(\cdot) = \|\cdot \mid X\|$ be a twice continuous (Frechet) differentiable functional on $X \setminus \{0\}$ and let $d^2N(\cdot)$ be the second Frechet differential of this norm. The norm of the extremal function $u_{j,\mathrm{opt}}^{(k)}$ for the optimal error functional $l_{j,\mathrm{opt}}^{(k)}$ is less than the norm of an arbitrary element of $V_j^{(k)}$;*

$$u_{j,\mathrm{opt}}^{(k)}(x) = \arg\min\{\|v \mid X\| \mid v \in V_j^{(k)}\}. \tag{2.11}$$

*There is a unique element of $V_j^{(k)}$ with this property.*

*Proof.* Let $v \in V_j^{(k)}$, $v \neq u_{j,\mathrm{opt}}^{(k)}$, and $0 \le t \le 1$. The function $\varphi(t) = \|(1-t)u_{j,\mathrm{opt}}^{(k)} + tv \mid X\|$ is twice differentiable and, by the Taylor formula, we have

$$\varphi(t) = \varphi(0) + t\varphi'(0) + \frac{1}{2}\varphi''(\tau), \quad \tau \in [0, t], \tag{2.12}$$

where $\varphi'(0) = (N'(u_{j,\mathrm{opt}}^{(k)}), v - u_{j,\mathrm{opt}}^{(k)})$ and

$$\varphi''(\tau) = d^2N(u_{j,\mathrm{opt}}^{(k)} + \tau(v - u_{j,\mathrm{opt}}^{(k)}))(v - u_{j,\mathrm{opt}}^{(k)}, v - u_{j,\mathrm{opt}}^{(k)}).$$

Using (2.12) together with (2.4), we obtain

$$\varphi'(0) = (N'(u^{(k)}_{j,\mathrm{opt}}), v - u^{(k)}_{j,\mathrm{opt}}) = \frac{1}{\|l^{(k)}_{j,\mathrm{opt}} \mid Y\|}(l^{(k)}_{j,\mathrm{opt}}, v - u^{(k)}_{j,\mathrm{opt}}).$$

Since $v - u^{(k)}_{j,\mathrm{opt}}$ belongs to $M^{(k)}_j$ and $M^{(k)}_j$ is embedded into the kernel of $l^{(k)}_{j,\mathrm{opt}}$, it follows that $\varphi'(0) = 0$.

By a definition, the second Frechet differential $d^2N$ of $N(\cdot)$ is a continuous symmetric bilinear function on $X \times X$. Considering that $N(\cdot)$ is convex on $X$, we show that $d^2N$ is positive semi-definite for every $v \in X$.

Let $u \in X$, $v \in X$, and

$$f(t) = (1-t)\|u\| + t\|v\| - \|(1-t)u + tv\|.$$

It is evident that $f(t) \geq 0$ and $f(0) = f(1) = 0$. Hence $f'(0) \geq 0$. Applying the Taylor formula to $f(t)$, we derive that

$$f(1) = f(0) + f'(0) + \frac{1}{2}f''(\tau) \quad \Longleftrightarrow \quad -\frac{1}{2}f''(\tau) = f'(0) \geq 0,$$

where $\tau \in [0,1]$. Since $f''(\tau) = -d^2N(u + \tau(v-u))(v-u, v-u)$, we have

$$d^2N(u + \tau(v-u))(v-u, v-u) \geq 0.$$

Putting in (2.12) $t = 1$, we observe that

$$\|v \mid X\| = \|u^{(k)}_{j,\mathrm{opt}} \mid X\| + \frac{1}{2}d^2N(u^{(k)}_{j,\mathrm{opt}} + \tau(v - u^{(k)}_{j,\mathrm{opt}}))(v - u^{(k)}_{j,\mathrm{opt}}, v - u^{(k)}_{j,\mathrm{opt}}).$$

Hence $\|v \mid X\| \geq \|u^{(k)}_{j,\mathrm{opt}} \mid X\|$, and (2.11) holds.

Since $V^{(k)}_j$ is a closed convex subset of $X$, it follows that $V^{(k)}_j$ is a Chebyshev set, and there is a unique function with (2.11). □

## 3. Extremal Functions and Splines of Affine Varieties

Let $M$ be a closed linear subspace of $X$. Assume that $M$ has finite codimension in $X$. Given an element $u \in X$, we consider $V = u + M$ a flat parallel to $M$.

**Definition 3.** [6, p. 576] *Let $u_0 \in X$, and*

$$u_0 = \arg\min\{\|v \mid X\| \mid v \in V\}.$$

*Then $u_0$ is said to be a spline of $V$.*

Following [6], we can now consider *the spline operator* $S : X \to X$ *of* $M$. By a definition, for every $u \in X$ $S(u)$ is the spline of $u + M$. Whence and from Definition 3 it follows that $S = S(M)$ is the mapping $I - P_M$, where $I$ is the identity map on $X$ and $P_M$ is the metric projection of $X$ onto $M$. Some authors use for the metric projection $P_M$ the term *normal projection*, or *best approximation operator*, or *nearest point map*, or *Chebyshev map*. By a definition, for every $u \in X$ $P_M(u)$ is the element of best approximation to $u$ from $M$. The spline operator $S$ of $M$ is homogeneous, i.e. $S(\lambda u) = \lambda S(u)$ for $\forall\, \lambda \in R$. The spline operator $S$ is linear (resp. continuous) iff $P_M$ is linear (resp. continuous). If $X$ is a Hilbert space, then $P_M$ and $S$ are linear and continuous. The linearity of metric projections is an infrequent phenomenon in non-Hilbert spaces. There are examples of linear metric projections in non-Hilbert spaces (see, e.g., [6, p. 580]).

Throughout the sequel we are interested in the spline operators of the following sequence of the closed linear subspaces of $X$

$$M_j^{(k)}= \{v \in X \mid (l_1^{(0)}, v) = 0,\ (\delta(x - \tilde{x}_i^{(m)}), v) = 0\, \forall\, \tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}\}.$$

We are also interested in the splines of flats $\tilde{V}_j^{(k)}=\overline{u}_{j,\mathrm{opt}}^{(k)}+ M_j^{(k)}$ where $\overline{u}_{j,\mathrm{opt}}^{(k)} = \frac{1}{\|u_{j,\mathrm{opt}}^{(k)}|X\|^2} u_{j,\mathrm{opt}}^{(k)}$. Evidently $M_j^{(k)}$ has the finite codimension. The spline operator of $M_j^{(k)}$ is denoted by $S_j^{(k)}$; $S_j^{(k)} = I - P_{M_j^{(k)}}$. Let us show the validity of the following

**Lemma 3.** *Let* $k < k_1$ *or* ($k = k_1$ *and* $j \le j_1$). *Then the equality holds*

$$S_{j_1}^{(k_1)} S_j^{(k)} = S_j^{(k)}. \tag{3.1}$$

*If the norm* $N(\cdot) = \|\cdot \mid X\|$ *is a twice continuous (Frechet) differentiable functional on* $X \setminus \{0\}$ *then* $S_j^{(k)}(u_{j,\mathrm{opt}}^{(k)}) = u_{j,\mathrm{opt}}^{(k)}$.

*Proof.* Since $S_{j_1}^{(k_1)} S_j^{(k)} = S_j^{(k)} - P_{M_{j_1}^{(k_1)}} S_j^{(k)}$, it follows that (3.1) holds iff

$$\forall\, u \in X \quad S_j^{(k)}(u) \in \ker\, P_{M_{j_1}^{(k_1)}}. \tag{3.2}$$

By Definition 3, $\ker\, P_{M_{j_1}^{(k_1)}} = \{w \in X \mid w = S_{j_1}^{(k_1)}(w)\}$, and $w$ belongs to the kernel of $P_{M_{j_1}^{(k_1)}}$ iff

$$\forall\, v \in M_{j_1}^{(k_1)} \quad (N'(w), v) = 0. \tag{3.3}$$

Since $S_j^{(k)}(u)$ is the spline of $u+M_j^{(k)}$, it follows from Corollary 3.1 [6, p. 584] that for all $v \in M_j^{(k)}$ $(N'(S_j^{(k)}(u)), v) = 0$. Using this equality, together with the fact that for $k < k_1$ or ($k = k_1$ and $j \leq j_1$) $M_{j_1}^{(k_1)}$ is a subset of $M_j^{(k)}$, we arrive at the sought relation (3.3) where $w = S_j^{(k)}(u)$. Consequently (3.2) is also valid.

If the norm $N(\cdot) = \|\cdot \mid X\|$ is a twice continuous (Frechet) differentiable functional on $X \setminus \{0\}$ then (2.11) holds. Considering that $\overline{u}_{j,\mathrm{opt}}^{(k)} \in \tilde{V}_j^{(k)}$ and using the homogeneity of $S_j^{(k)}$, we obtain

$$\overline{u}_{j,\mathrm{opt}}^{(k)} = \arg\min\{\|v \mid X\| \mid v \in \tilde{V}_j^{(k)}\} = S_j^{(k)}(\overline{u}_{j,\mathrm{opt}}^{(k)}).$$

Finally, by the homogeneity of $S_j^{(k)}$, we come to $S_j^{(k)}(u_{j,\mathrm{opt}}^{(k)}) = u_{j,\mathrm{opt}}^{(k)}$. □

## 4. Multigrid and Error Multifunctional with Agreement Conditions

Throughout the sequel we assume that
**(S)** *the optimal error multifunctional consists of the pairwise distinct functionals.*
This assumption is not superfluous. An example is as follows.

Let $\Omega$ be the unit ball of $R^n$ and let $X$ be a space of harmonic functions with domain $\Omega$. If members of $X$ are continuous functions in the closure of $\Omega$, then Mean Value Theorem implies that

$$(\chi_\Omega(x), u(x)) = (\frac{1}{|\Omega|}\delta(x), u(x)) \quad \text{for} \quad \forall\, u \in X.$$

Now take an arbitrary multigrid $\boldsymbol{\Delta}$ and assume that $\tilde{x}_1^{(0)} = 0$. In this event each optimal error functional is identically 0.

In general the initial functional $l_1^{(0)}$ does not agree with a linear combination of Dirac delta functions, and the hypothesis **(S)** seems to be very natural. We now dwell in more detail on the explanation of this claim.

Let $\mathrm{Nod}_j^{(k)} \subset \mathrm{Nod}_{j_1}^{(k_1)}$. Then $l_j^{(k)}$ is an error functional with nodes in $\mathrm{Nod}_{j_1}^{(k_1)}$; and the weights of $l_j^{(k)}$ at the points of $\mathrm{Nod}_{j_1}^{(k_1)} \setminus \mathrm{Nod}_j^{(k)}$ equal 0. Hence, $\|l_{j_1,\mathrm{opt}}^{(k_1)} \mid X^*\| \leq \|l_{j,\mathrm{opt}}^{(k)} \mid X^*\|$. If $l_{j_1,\mathrm{opt}}^{(k_1)} = l_{j,\mathrm{opt}}^{(k)}$, then

$$\|l_{j,\mathrm{opt}}^{(k)} \mid Y\| = \|l_{j+1,\mathrm{opt}}^{(k)} \mid Y\| = \cdots = \|l_{j_1-1,\mathrm{opt}}^{(k_1)} \mid Y\| = \|l_{j_1,\mathrm{opt}}^{(k_1)} \mid Y\|.$$

Since the optimal cubature formula with the given set $\mathrm{Nod}_{j_1}^{(k_1)}$ of nodes is unique it follows that

$$l_{j,\mathrm{opt}}^{(k)} = l_{j+1,\mathrm{opt}}^{(k)} = \cdots = l_{j_1-1,\mathrm{opt}}^{(k_1)} = l_{j_1,\mathrm{opt}}^{(k_1)}.$$

Thus we extend the set $\mathrm{Nod}_j^{(k)}$ of nodes to $\mathrm{Nod}_{j_1}^{(k_1)}$ and with it all the norm of the optimal error functional does not decrease. Hence some levels of the initial multigrid $\boldsymbol{\Delta}$ contain of a few "irrelevant" nodes and the hypothesis $(\mathbf{S})$ means that we do not consider a multigrid with the "irrelevant" nodes. In this event we say that $\boldsymbol{\Delta}$ and the optimal error multifunctional satisfy *the agreement conditions.* In particular, under assumption $(\mathbf{S})$ for $(k,j) \neq (k_1, j_1)$ we have

$$\text{(4.1)} \qquad \mathrm{Nod}_j^{(k)} \subset \mathrm{Nod}_{j_1}^{(k_1)} \implies \|l_{j_1,\mathrm{opt}}^{(k_1)} \mid Y\| < \|l_{j,\mathrm{opt}}^{(k)} \mid Y\|.$$

**Lemma 4.** *Assuming* $(\mathbf{S})$ *the extremal function* $u_{j,\mathrm{opt}}^{(k)}$ *for* $l_{j,\mathrm{opt}}^{(k)}$ *is not equal to 0 at* $\tilde{x}_j^{(k)}$.

*Proof.* Let $u_{j,\mathrm{opt}}^{(k)}(\tilde{x}_j^{(k)}) = 0$. Considering that $u_{j,\mathrm{opt}}^{(k)}$ equals 0 at the nodes of $\mathrm{Nod}_j^{(k)}$, we come to the conclusion that for $j \le \sigma(k) - 1$ the equalities hold

$$(l_{j+1,\mathrm{opt}}^{(k)}, u_{j,\mathrm{opt}}^{(k)}) = (l_1^{(0)}, u_{j,\mathrm{opt}}^{(k)}) = (l_{j,\mathrm{opt}}^{(k)}, u_{j,\mathrm{opt}}^{(k)}) = \|l_{j,\mathrm{opt}}^{(k)} \mid Y\|^2.$$

Further, dividing both sides of this chains of the equalities by the norm $\|u_{j,\mathrm{opt}}^{(k)} \mid X\| = \|l_{j,\mathrm{opt}}^{(k)} \mid Y\|$, we obtain

$$\|l_{j+1,\mathrm{opt}}^{(k)} \mid Y\| \ge \frac{(l_{j+1,\mathrm{opt}}^{(k)}, u_{j,\mathrm{opt}}^{(k)})}{\|u_{j,\mathrm{opt}}^{(k)} \mid X\|} = \|l_{j,\mathrm{opt}}^{(k)} \mid Y\|$$

which contradicts with (4.1). If $j = \sigma(k)$, then we must consider $l_{1,\mathrm{opt}}^{(k+1)}$ instead of $l_{j+1,\mathrm{opt}}^{(k)}$. $\square$

## 5. Constructing a $\boldsymbol{\Delta}$-Hierarchical Basis

Let $b_j^{(k)} = u_{j,\mathrm{opt}}^{(k)}(\tilde{x}_j^{(k)})$. By Lemma 4, $b_j^{(k)} \neq 0$, and we can deal with the function $\frac{1}{b_j^{(k)}} u_{j,\mathrm{opt}}^{(k)}(x)$. By Theorem 7, the function $\frac{1}{b_j^{(k)}} u_{j,\mathrm{opt}}^{(k)}$ is a member of the following affine variety

$$\{v \in X \mid (\delta(x - \tilde{x}_j^{(k)}), v) = 1;\ (\delta(x - \tilde{x}_i^{(m)}), v) = 0 \,\forall\, \tilde{x}_i^{(m)} \in \mathrm{Nod}_j^{(k)}\}.$$

Let $\tilde{u}^{(k)}_{j,\mathrm{opt}}$ be the spline of this affine variety and let $\tilde{S}^{(k)}_j$ be the corresponding spline operator, i.e.,

$$\tilde{u}^{(k)}_{j,\mathrm{opt}} = \tilde{S}^{(k)}_j\Big(\frac{1}{b^{(k)}_j} u^{(k)}_{j,\mathrm{opt}}\Big).$$

By the definition of $\tilde{S}^{(k)}_j$, the function $\tilde{u}^{(k)}_{j,\mathrm{opt}}$ agrees with the function $\frac{1}{b^{(k)}_j} u^{(k)}_{j,\mathrm{opt}}(x)$ at the nodes of $\mathrm{Nod}^{(k)}_j \cup \{\tilde{x}^{(k)}_j\}$. If $k < k_1$ or ($k = k_1$ and $j \le j_1$), then it follows in much the same way as in Lemma 3 that

$$\tilde{S}^{(k_1)}_{j_1}\tilde{S}^{(k)}_j = \tilde{S}^{(k)}_j; \quad \tilde{S}^{(k)}_j(\tilde{u}^{(k)}_{j,\mathrm{opt}}) = \tilde{u}^{(k)}_{j,\mathrm{opt}}. \tag{5.1}$$

**Lemma 5.** *For given integers $k$ and $j$ there is a unique function $h^{(k)}_j$ such that*

$$h^{(k)}_j \in \text{ span } \{\tilde{u}^{(k)}_{j,\mathrm{opt}}(x), \tilde{u}^{(k)}_{j+1,\mathrm{opt}}(x), \ldots, \tilde{u}^{(k)}_{\sigma(k),\mathrm{opt}}(x)\}$$

*and (1.1) holds. Assuming $\tilde{h}^{(k)}_j = \tilde{S}^{(k)}_{\sigma(k)}(h^{(k)}_j)$, the set*

$$H_{\boldsymbol{\Delta}} = \{\tilde{h}^{(k)}_j \mid k = 0, 1, \ldots; \, j = 1, 2, \ldots, \sigma(k)\}$$

*is a $\boldsymbol{\Delta}$-hierarchical system of $X$. If for an arbitrary $k$ the spline operator $\tilde{S}^{(k)}_{\sigma(k)}$ is linear then $h^{(k)}_j = \tilde{h}^{(k)}_j(x)$.*

*Proof.* Given $k$ and $j$ we seek a solution $h^{(k)}_j$ to (1.1) in the form of the linear combination of the splines $\tilde{u}^{(k)}_{j,\mathrm{opt}}$, $\tilde{u}^{(k)}_{j+1,\mathrm{opt}}$, ..., $\tilde{u}^{(k)}_{\sigma(k),\mathrm{opt}}$. To be more precise, we assume that $\tilde{u}^{(k)}_{\sigma(k)+1,\mathrm{opt}} = \tilde{u}^{(k+1)}_{1,\mathrm{opt}}$ for $k = 0, 1, 2, \ldots$, and suppose

$$h^{(k)}_j = \sum_{i=1}^{\sigma(k)-j+1} \alpha^{(j,k)}_i \tilde{u}^{(k)}_{j+i,\mathrm{opt}} - \tilde{u}^{(k)}_{j,\mathrm{opt}}. \tag{5.2}$$

Examine that there are $\alpha^{(j,k)}_i$, $i = 1, 2, \ldots, \sigma(k) - j + 1$, such that

$$\begin{aligned} &h^{(k)}_j(\tilde{x}^{(k)}_{j+l}) = 0 \quad \text{for} \quad l = 1, 2, \ldots, \sigma(k) - j, \\ &\sum_{i=1}^{\sigma(k)-j+1} \alpha^{(j,k)}_i = 1. \end{aligned}$$

By (2.7), we can write the last equalities in the equivalent form

$$\begin{aligned} \sum_{i=1}^{l} \alpha^{(j,k)}_i \tilde{u}^{(k)}_{j+i,\mathrm{opt}}(\tilde{x}^{(k)}_{j+l}) &= \tilde{u}^{(k)}_{j,\mathrm{opt}}(\tilde{x}^{(k)}_{j+l}), \\ & l = 1, 2, \ldots, \sigma(k) - j; \\ \sum_{i=1}^{\sigma(k)-j+1} \alpha^{(j,k)}_i &= 1. \end{aligned}$$

It is a system of linear equations with respect to unknown coefficients $(\alpha_1^{(j,k)}, \ldots, \alpha_{\sigma(k)-j+1}^{(j,k)})$. The matrix of this system is subdiagonal with 1 on the main diagonal. Hence, this matrix is non-singular; and there exists a unique solution $(\alpha_1^{(j,k)}, \ldots, \alpha_{\sigma(k)-j+1}^{(j,k)})$ to the system under consideration.

Thus, function $h_j^{(k)}$ is uniquely determined, and it is easy to show that (1.1) holds and $H_{\boldsymbol{\Delta}}$ is actually a $\boldsymbol{\Delta}$-hierarchical system of $X$.

Let $\tilde{S}_{\sigma(k)}^{(k)}$, $k = 0, 1, 2, \ldots$, be linear spline operators. By the definition of $h_j^{(k)}$ it follows from (5.1) that

$$h_j^{(k)} = \sum_{i=1}^{\sigma(k)-j+1} \alpha_i^{(j,k)} \tilde{S}_{j+i}^{(k)}(\tilde{u}_{j+i,\mathrm{opt}}^{(k)}) - \tilde{S}_j^{(k)}(\tilde{u}_{j,\mathrm{opt}}^{(k)}).$$

Applying $\tilde{S}_{\sigma(k)}^{(k)}$ to both sides of the last equality and using (5.1) again, we arrive at the sought equality $\tilde{h}_j^{(k)} = h_j^{(k)}$. $\square$

**Theorem 9.** *If there is a Frechet differential of $\|\cdot \mid Y\|$ at $l \in Y$, $l \neq 0$, and the spline operators $\tilde{S}_{\sigma(k)}^{(k)}$, $k = 0, 1, \ldots$, are linear then*

$$H_{\boldsymbol{\Delta}} = \{\tilde{h}_j^{(k)}(x) \mid k = 0, 1, \ldots; \; j = 1, 2, \ldots, \sigma(k)\}$$

*is a $\boldsymbol{\Delta}$-hierarchical basis of $X$.*

*Proof.* Since the spline operators $\tilde{S}_{\sigma(k)}^{(k)}$, $k = 0, 1, \ldots$, are linear it follows from Lemma 5 that $\tilde{h}_j^{(k)} = h_j^{(k)}$. Hence it is sufficient to expand an arbitrary function $\varphi \in X$ in the series with respect to $h_j^{(k)}$ and check the convergence of this series in the norm of $X$.

For a given function $\varphi \in X$ and a positive integer $m$ there are coefficients $g_j^{(k)}$ of the sum $\sigma_m(x) = \sum\limits_{k=0}^{m} \sum\limits_{j=1}^{\sigma(k)} g_j^{(k)} h_j^{(k)}(x)$ such that $\sigma_m(x) = \varphi(x)$ for $\forall\, x \in \Delta_m$. Considering that $H_{\boldsymbol{\Delta}}$ is a $\boldsymbol{\Delta}$-hierarchical system by easy calculations we obtain the following recurrent relations for $g_j^{(k)}$ (see [15] and [16])

$$\begin{aligned} g_j^{(0)} &= \sigma_m(\tilde{x}_j^{(0)}) = \varphi(\tilde{x}_j^{(0)}), \; j = 1, 2, \ldots, \sigma(0); \\ \varphi_0(x) &= \sum_{j=1}^{\sigma(0)} g_j^{(0)} h_j^{(0)}(x), \end{aligned}$$

and further for $k = 1, 2, \ldots, m$

$$\begin{aligned} g_j^{(k)} &= \sigma_m(\tilde{x}_j^{(k)}) - \varphi_{k-1}(\tilde{x}_j^{(k)}) = \varphi(\tilde{x}_j^{(k)}) - \varphi_{k-1}(\tilde{x}_j^{(k)}), \\ &\quad j = 1, 2, \ldots, \sigma(k), \\ \varphi_k(x) &= \varphi_{k-1}(x) + \sum_{j=1}^{\sigma(k)} g_j^{(k)} h_j^{(k)}(x). \end{aligned}$$

The coefficients $g_j^{(k)}$ are independent of $m$. Hence the function $\sigma_m(x)$ is a partial sum of the series $\sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(k)} g_j^{(k)} h_j^{(k)}$. This series is convergent to $\varphi(x)$ in the norm of $X$ iff $\lim_{m\to\infty} \|\varphi - \sigma_m \mid X\| = 0$. Let us show that it is a valid equality.

By the definition of $\sigma_m(x)$ and from the property of $\tilde{S}_{\sigma(m)}^{(m)}$ it follows that $\tilde{S}_{\sigma(m)}^{(m)}(\varphi) = \tilde{S}_{\sigma(m)}^{(m)}(\sigma_m)$. The linearity of $\tilde{S}_{\sigma(m)}^{(m)}$ together with Lemma 5 and equalities (5.1) imply that

$$\begin{aligned} \tilde{S}_{\sigma(m)}^{(m)}(\sigma_m) &= \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} g_j^{(k)} \tilde{S}_{\sigma(m)}^{(m)}(h_j^{(k)}) = \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} g_j^{(k)} \tilde{S}_{\sigma(m)}^{(m)} \tilde{S}_{\sigma(k)}^{(k)}(h_j^{(k)}) \\ &= \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} g_j^{(k)} \tilde{S}_{\sigma(k)}^{(k)}(h_j^{(k)}) = \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} g_j^{(k)} h_j^{(k)} = \sigma_m. \end{aligned}$$

We now show that $\lim_{m\to\infty} \|\varphi - \sigma_m \mid X\| = \lim_{m\to\infty} \|\varphi - \tilde{S}_{\sigma(m)}^{(m)}(\varphi) \mid X\| = 0$.

Let $M^{(m)} = \{v \in X \mid v(\tilde{x}_s^{(p)}) = 0 \,\forall\, \tilde{x}_s^{(p)} \in \Delta_m\}$ and

$$V^{(m)}(\varphi) = \varphi + M^{(m)}.$$

Then $V^{(m+1)}(\varphi) \subset V^{(m)}(\varphi)$. Since $\varphi \in \bigcap_{m=0}^{\infty} V^{(m)}(\varphi)$ it follows that for $m = 0, 1, \ldots$ the inequalities hold

$$\|\tilde{S}_{\sigma(m)}^{(m)}(\varphi) \mid X\| \le \|\tilde{S}_{\sigma(m+1)}^{(m+1)}(\varphi) \mid X\| \le \|\varphi \mid X\|. \tag{5.3}$$

Hence the bounded sequence $\{\|\tilde{S}_{\sigma(m)}^{(m)}(\varphi) \mid X\|\}_{m=0}^{\infty}$ of norms increases to a finite limit $A$;

$$\lim_{m\to\infty} \|\tilde{S}_{\sigma(m)}^{(m)}(\varphi) \mid X\| = A \le \|\varphi \mid X\| < \infty. \tag{5.4}$$

Because $X$ is reflexive, there is a weak convergent subsequence of $\{\tilde{S}_{\sigma(m)}^{(m)}(\varphi)\}_{m=0}^{\infty}$. Let $\overline{\varphi}$ be the weak limit of the subsequence; $\overline{\varphi} \in X$. For simplicity and without loss of generality, we assume that

$\{\tilde{S}^{(m)}_{\sigma(m)}(\varphi)\}_{m=0}^{\infty}$ is weakly convergent to $\overline{\varphi}$. Then for $\forall \tilde{x}_j^{(k)} \in \bigcup_{m=0}^{\infty} \Delta_m$ the equalities hold

$$\overline{\varphi}(\tilde{x}_j^{(k)}) = (\delta(x - \tilde{x}_j^{(k)}), \overline{\varphi}) = \lim_{m\to\infty} (\delta(x - \tilde{x}_j^{(k)}), \tilde{S}^{(m)}_{\sigma(m)}(\varphi)) = \varphi(\tilde{x}_j^{(k)}).$$

The difference $\varphi - \overline{\varphi}$ is continuous in $\overline{\Omega}$ and equals 0 at the nodes of $\bigcup_{m=0}^{\infty} \Delta_m$. By the assumption, the set $\bigcup_{m=0}^{\infty} \Delta_m$ is dense in $\overline{\Omega}$. Hence $\varphi - \overline{\varphi}$ is identically equal to 0 in $\overline{\Omega}$; and the weak limit of the spline sequence $\{\tilde{S}^{(m)}_{\sigma(m)}(\varphi)\}_{m=0}^{\infty}$ coincides with $\varphi$. Let us show that splines $\tilde{S}^{(m)}_{\sigma(m)}(\varphi)$ converge to $\varphi$ in the norm of $X$.

Since $\{\tilde{S}^{(m)}_{\sigma(m)}(\varphi)\}_{m=0}^{\infty}$ is weakly convergent to $\varphi$ it follows from (5.4) that $\pi^*(\varphi)$ satisfies

$$\|\varphi \mid X\|^2 = (\pi^*(\varphi), \varphi) = \lim_{m\to\infty} (\pi^*(\varphi), \tilde{S}^{(m)}_{\sigma(m)}(\varphi))$$

$$\leq \|\pi^*(\varphi) \mid Y\| \lim_{m\to\infty} \|\tilde{S}^{(m)}_{\sigma(m)}(\varphi) \mid X\| \leq A\|\varphi \mid X\|.$$

Consequently, $\|\varphi \mid X\| \leq A$. This estimation together with (5.4) implies

$$\|\varphi \mid X\| = A = \lim_{m\to\infty} \|\tilde{S}^{(m)}_{\sigma(m)}(\varphi) \mid X\|. \tag{5.5}$$

Put $v_m = \frac{1}{\|\tilde{S}^{(m)}_{\sigma(m)}(\varphi)|X\|}\tilde{S}^{(m)}_{\sigma(m)}(\varphi)$. Then $\|v_m \mid X\| = 1$. Applying (5.5), it is not hard to validate that $\{v_m\}_{m=0}^{\infty}$ converges weakly to $\frac{1}{\|\varphi|X\|}\varphi$. In particular, the equalities hold

$$\lim_{m\to\infty} (\frac{1}{\|\pi^*(\varphi) \mid X\|}\pi^*(\varphi), v_m) = \frac{1}{\|\varphi \mid X\|\|\pi^*(\varphi) \mid X\|}(\pi^*(\varphi), \varphi) = 1.$$

Given the member $l = \pi^*(\frac{1}{\|\pi^*(\varphi)|X\|}\varphi)$ of the unit sphere of $Y$, we have pointed out the sequence $\{v_m\}_{m=0}^{\infty}$ of members of the unit sphere of $X = Y^*$ such that $\lim\limits_{m\to\infty} (l, v_m) = 1$. By the hypothesis, there is a Frechet differential $\Phi = N_G^{*,\prime}(l)$ of the norm $\|\cdot \mid Y\|$ at $l \in Y$. Considering that (2.6) holds we have $N_G^{*,\prime}(l) = N_G^{*,\prime}(\pi^*(\frac{1}{\|\varphi|X\|}\varphi)) = \frac{1}{\|\varphi|X\|}\varphi$. By the Shmulian criterion [7, p. 147], $\{v_m\}_{m=0}^{\infty}$ must converge to $\Phi = \frac{1}{\|\varphi|X\|}\varphi$ in the norm of $X$. The proof of the convergence in more detail is as follows.

If $\|v_m - \Phi\| \nrightarrow 0$, then $\exists\, \varepsilon > 0$ and $\{l_m\}_{m=0}^{\infty} \subset Y$ such that $\|l_m \mid Y\| = 1$ and $(l_m, v_m - \Phi) \geq 2\varepsilon$ for $\forall\, m \geq 1$. Let

$$\tilde{l}_m = \frac{1}{\varepsilon}(\|l \mid Y\| - (l, v_m))l_m = \|\tilde{l}_m \mid Y\| l_m.$$

Then $\|\tilde{l}_m \mid Y\| = \frac{1}{\varepsilon}(1 - (l, v_m)) \to 0$ and

$$\frac{\|l + \tilde{l}_m \mid Y\| - \|l \mid Y\| - (\tilde{l}_m, \Phi)}{\|\tilde{l}_m \mid Y\|} \geq \frac{(l + \tilde{l}_m, v_m) - 1 - (\tilde{l}_m, \Phi)}{\|\tilde{l}_m \mid Y\|}$$

$$= \frac{(l, v_m) - 1 + \|\tilde{l}_m \mid Y\|(l_m, v_m - \Phi)}{\|\tilde{l}_m \mid Y\|} = -\varepsilon + (l_m, v_m - \Phi) \geq \varepsilon,$$

which contradicts that $\Phi$ is a Frechet differential of $\|\cdot \mid Y\|$ at $l \in Y$.

Finally, the equality holds $\lim\limits_{m \to \infty} \|v_m - \Phi\| = 0$. Whence and from (5.5) the spline sequence $\{\tilde{S}_{\sigma(m)}^{(m)}(\varphi)\}_{m=0}^{\infty}$ converges to $\varphi$ in the norm of $X$. $\square$

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