Selçuk J. Appl. Math.
Vol. 2, No. 2, pp. 39–52, 2001

Selçuk Journal of Applied Mathematics

On a functional approach to spectral problems of linear algebra

Gennadiĭ Demidenko*

Sobolev Institute of Mathematics, SB RAS, 630090 Novosibirsk, Russia;
e-mail: demidenk@math.nsc.ru

Received: October 16, 2001

Summary. In the paper we present some applications of a new method for constructing approximate projections onto invariant subspaces of linear operators. We illustrate the method on the dichotomy problem for the matrix spectrum with respect to an ellipse. We prove Lyapunov type theorems on location of the matrix spectrum with respect to an ellipse.

Key words: projections onto invariant subspaces, matrix spectrum dichotomy, matrix equations

Mathematics Subject Classification (1991): 15A60, 47A15, 47A58, 65F30

1. Introduction

The paper is devoted to constructing projections onto invariant subspaces of matrices. The problem of constructing projections is a very important problem of the theory of linear operators, it has numerous applications. At present, there are some algorithms for constructing projections: for example, classical results by J. von Neumann, F. Riesz, M. G. Krein (see, for example, [1, 2]), the well-known matrix sign function method [3, 4], algorithms with guaranteed accuracy

* The research was financially supported by the Scientific and Technical Research Council of Turkey (TUBITAK) in the framework of the NATO-PC Advanced Fellowships Programme.

for solving dichotomy problems for the matrix spectrum [5, 6] and so on.

In 1996, the author [7] proposed a new method for constructing approximate projections onto invariant subspaces of linear operators. The method has applications in linear algebra for constructing projections onto invariant subspaces of matrices. In particular, using this method, one can obtain formulae for approximate projections in dichotomy and trichotomy problems for the matrix spectrum (see [8, 9]). In the paper we use this method for constructing approximate projections in the dichotomy problem for the matrix spectrum with respect to an ellipse. We prove also Lyapunov type theorems on location of the matrix spectrum with respect to an ellipse.

In our opinion, the elliptic dichotomy problem can have important applications to numerical research of spectral portraits of matrices (see, for example, [6, 10–12]).

2. Basic ideas of the method for constructing approximate projections

The proposed method [7 - 9] is functional and based on the following theorems by the author.

Theorem 1. *Let $T : B \to B$ be a linear continuous operator in a Banach space B, and let T have the inverse operator T^{-1}. Suppose that there is a projection $P : B \to B$ such that $PT = TP$, and the following estimates hold*

$$\|TP\| < 1, \quad \|T^{-1}(I-P)\| < 1.$$

Then the operator $I - T$ has the continuous inverse one $(I-T)^{-1}$, and the representation

$$(I-T)^{-1} = (I-TP)^{-1}P - (I - T^{-1}(I-P))^{-1}T^{-1}(I-P)$$

is true.

Corollary 1. *The inverse operator $(I-T)^{-1}$ can be rewritten as follows*

$$(I-T)^{-1} = P + TP(I-TP)^{-1} - T^{-1}(I-P)(I-T^{-1}(I-P))^{-1}.$$

Corollary 2. *The inequality*

$$\|(I-T)^{-1} - P\| \le \|TP\|(1-\|TP\|)^{-1}$$
$$+\|T^{-1}(I-P)\|(1-\|T^{-1}(I-P)\|)^{-1}$$

is satisfied.

Remark 1. From this corollary we have the following important result. If the norms $||TP||$, $||T^{-1}(I-P)||$ are sufficiently small:

$$||TP|| \approx 0, \qquad ||T^{-1}(I-P)|| \approx 0,$$

then the inverse operator $(I-T)^{-1}$ is close to the projection P:

$$(I-T)^{-1} \approx P.$$

Theorem 2. *Let*

$$T : B \to B, \qquad \widetilde{T} : B \to B$$

be linear continuous operators in a Banach space B and let the operator T satisfy the conditions of Theorem 1, i.e. T has the inverse operator T^{-1} and there is a projection

$$P : B \to B$$

such that the following conditions hold

$$PT = TP, \qquad ||TP|| < 1, \qquad ||T^{-1}(I-P)|| < 1.$$

If the condition

$$||(\widetilde{T}-T)(I-T)^{-1}|| < 1$$

is satisfied, then the operator $I-\widetilde{T}$ has the inverse one $(I-\widetilde{T})^{-1}$, and the estimate holds

$$||(I-\widetilde{T})^{-1} - P||$$

$$\leq c(||TP||, ||T^{-1}(I-P)||, ||(\widetilde{T}-T)(I-T)^{-1}||),$$

where $c(x,y,z)$ is a continuous function with $c(0,0,0)=0$.

Remark 2. From Theorem 2 we have the following result. If the norms $||TP||$, $||T^{-1}(I-P)||$, $||(\widetilde{T}-T)(I-T)^{-1}||$ are sufficiently small:

$$||TP|| \approx 0, \quad ||T^{-1}(I-P)|| \approx 0, \quad ||(\widetilde{T}-T)(I-T)^{-1}|| \approx 0,$$

then the inverse operator $(I-\widetilde{T})^{-1}$ is close to the projection P:

$$(I-\widetilde{T})^{-1} \approx P.$$

Remark 3. By Theorem 1, one can obtain a modification of the matrix sign function method (see [8, 9]).

3. The dichotomy problem for the matrix spectrum with respect to an ellipse

In the papers [8, 9] we illustrated applications of Theorems 1 and 2 to some spectral problems of linear algebra. In particular, we constructed approximate projections onto invariant subspaces of matrices for the dichotomy and trichotomy problems for the matrix spectrum with respect to the imaginary axis and a circle.

In this Section we illustrate applications of Theorems 1 and 2 to the problem of constructing approximate projections onto invariant subspaces of matrices in the dichotomy problem for the matrix spectrum with respect to an ellipse.

Let A be a square $N \times N$ matrix, let A^* be adjoint to the matrix A, and let $||A||$ be the spectral norm of A, i.e.,

$$||A|| = \max_{||u||=1} ||Au||,$$

where $||u|| = (\sum_{i=1}^{N} |u_i|^2)^{1/2}$ is the norm of a vector $u = (u_1, ..., u_N)$ from the space E_N. By

$$\langle u, v \rangle = \sum_{i=1}^{N} u_i \bar{v}_i$$

we denote the scalar product of two vectors $u, v \in E_N$.

Consider the dichotomy problem for the matrix spectrum with respect to the ellipse

$$\mathcal{E}_{\mathbf{o}} = \{\lambda \in C : \frac{(\mathrm{Re}\,\lambda)^2}{a^2} + \frac{(\mathrm{Im}\,\lambda)^2}{b^2} = 1\}, \quad a > b.$$

Condition. We suppose that the matrix A has no eigenvalues on the ellipse $\mathcal{E}_{\mathbf{o}}$ (fig. 1). Eigenvalues of the matrix A are unknown.

Introduce the following notation.

By $P_{\mathbf{i}}$ we denote the projection onto the maximal invariant subspace of the matrix A corresponding to the eigenvalues lying in

$$\mathcal{E}_{\mathbf{i}} = \{\lambda \in C : \frac{(\mathrm{Re}\,\lambda)^2}{a^2} + \frac{(\mathrm{Im}\,\lambda)^2}{b^2} < 1\},$$

and $AP_{\mathbf{i}} = P_{\mathbf{i}}A$.

By $P_{\mathbf{e}}$ we denote the projection onto the maximal invariant subspace of the matrix A corresponding to the eigenvalues lying in

$$\mathcal{E}_{\mathbf{e}} = \{\lambda \in C : \frac{(\mathrm{Re}\,\lambda)^2}{a^2} + \frac{(\mathrm{Im}\,\lambda)^2}{b^2} > 1\},$$

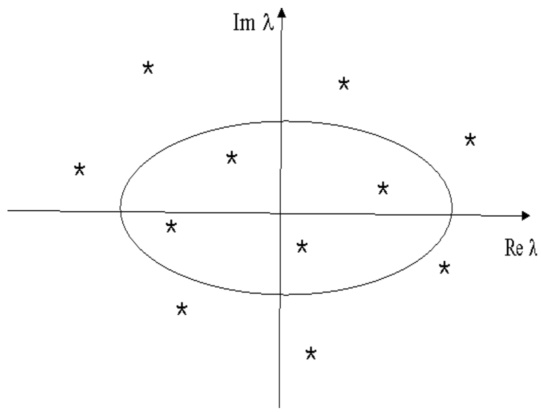

Fig. 1.

and $AP_{\mathbf{e}} = P_{\mathbf{e}}A$,

$$P_{\mathbf{i}} + P_{\mathbf{e}} = I.$$

Problem. Our aim is to construct approximate projections for the projections $P_{\mathbf{i}}$, $P_{\mathbf{e}}$.

Note that one variant for solving this problem was obtained by S. K. Godunov and M. Sadkane [13]. Using Theorems 1 and 2, we propose two news variants of solving this problem.

We will consider the case when the matrices

$$A - cI, \quad A + cI, \quad c^2 = a^2 - b^2,$$

are nonsingular, i. e. the focuses of the ellipse $\mathcal{E}_{\mathbf{o}}$ are not eigenvalues of the matrix A.

The first variant of solving the elliptic dichotomy problem

Consider the matrix sequence $\{T_k\}$:

$$T_k(A) = \frac{1}{(a+b)^k}(A + \sqrt{A^2 - c^2 I})^k, \quad k = 1, 2, \dots \tag{1}$$

Since the matrix A has no eigenvalues on the ellipse $\mathcal{E}_{\mathbf{o}}$ and its focuses are not eigenvalues of A, then there exist inverse matrices

$$T_k^{-1}(A) = \frac{1}{(a-b)^k}(A - \sqrt{A^2 - c^2 I})^k. \tag{2}$$

By $\lambda_j(B)$ we denote the jth eigenvalue of an $N \times N$ matrix B. Then eigenvalues of matrices $T_k(A)$ and $T_k^{-1}(A)$ have the form

$$\lambda_j(T_k(A)) = \frac{1}{(a+b)^k}(\lambda_j(A) + \sqrt{(\lambda_j(A))^2 - c^2})^k, \tag{3}$$

$$\lambda_j(T_k^{-1}(A)) = \frac{1}{(a-b)^k}(\lambda_j(A) - \sqrt{(\lambda_j(A))^2 - c^2})^k, \tag{4}$$

$$j = 1, \dots, N.$$

Hence, if

$$\lambda_p(A) \in \mathcal{E}_{\mathbf{i}}, \quad \lambda_q(A) \in \mathcal{E}_{\mathbf{e}},$$

then

$$\lambda_p(T_k(A)),\ \lambda_q(T_k^{-1}(A)) \in \{\lambda \in C : |\lambda| < 1\}.$$

From the definition of the projection $P_{\mathbf{i}}$ we have the convergences

$$\|T_k(A)P_{\mathbf{i}}\| \to 0, \quad \|T_k^{-1}(A)(I - P_{\mathbf{i}})\| \to 0, \quad k \to \infty.$$

By Theorem 1, there exist inverse matrices

$$(I - T_k(A))^{-1}$$

for sufficiently large $k \gg 1$, and the convergence

$$\|(I - T_k(A))^{-1} - P_{\mathbf{i}}\| \to 0, \quad k \to \infty,$$

holds. Hence, we obtain the approximate projections:

$$P_{\mathbf{i}} \approx (I - T_k(A))^{-1}, \quad P_{\mathbf{e}} \approx I - (I - T_k(A))^{-1}, \quad k \gg 1.$$

The second variant of solving the elliptic dichotomy problem

The use of the square root $\sqrt{A^2 - c^2 I}$ is not convenient for computation of approximate projections. Now we present another variant for constructing approximate projections for $P_{\mathbf{i}}$, $P_{\mathbf{e}}$.

By (1) and (2), it follows that

$$\frac{a+b}{2}T_1(A) + \frac{a-b}{2}T_1^{-1}(A) - A = 0.$$

For brevity, we will write $T_1 = T_1(A)$. Obviously, we have

$$T_1^2 - \frac{2}{a+b}AT_1 + \frac{a-b}{a+b}I = 0$$

and

$$((\frac{a+b}{a-b}T_1)^{-1})^2 - \frac{2}{a+b}A(\frac{a+b}{a-b}T_1)^{-1} + \frac{a-b}{a+b}I = 0.$$

Introduce the matrix

$$\mathcal{L} = \begin{pmatrix} 0 & I \\ -\frac{a-b}{a+b}I & \frac{2}{a+b}A \end{pmatrix}.$$

Then the above equalities can be rewritten as follows

$$\begin{pmatrix} T_1 \\ T_1^2 \end{pmatrix} = \mathcal{L} \begin{pmatrix} I \\ T_1 \end{pmatrix}, \quad \begin{pmatrix} \widetilde{T}_1 \\ \widetilde{T}_1^2 \end{pmatrix} = \mathcal{L} \begin{pmatrix} I \\ \widetilde{T}_1 \end{pmatrix}$$

or

$$\begin{pmatrix} T_1 & \widetilde{T}_1 \\ T_1^2 & \widetilde{T}_1^2 \end{pmatrix} = \mathcal{L} \begin{pmatrix} I & I \\ T_1 & \widetilde{T}_1 \end{pmatrix},$$

where

$$\widetilde{T}_1 = (\frac{a+b}{a-b} T_1)^{-1}.$$

Since

$$\begin{pmatrix} T_1 & \widetilde{T}_1 \\ T_1^2 & \widetilde{T}_1^2 \end{pmatrix} = \begin{pmatrix} I & I \\ T_1 & \widetilde{T}_1 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & \widetilde{T}_1 \end{pmatrix},$$

we have

$$\begin{pmatrix} I & I \\ T_1 & \widetilde{T}_1 \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & \widetilde{T}_1 \end{pmatrix} = \mathcal{L} \begin{pmatrix} I & I \\ T_1 & \widetilde{T}_1 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} T_1 & 0 \\ 0 & \widetilde{T}_1 \end{pmatrix} = \mathcal{V}^{-1} \mathcal{L} \mathcal{V}, \quad \mathcal{V} = \begin{pmatrix} I & I \\ T_1 & \widetilde{T}_1 \end{pmatrix}. \tag{5}$$

Using properties of the Zhukovskii function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

in the complex plane (see, for example, [14]) and the conditions for the matrix A, one can prove that the matrix $\mathcal{L}$ has no eigenvalues on the unit circle. Indeed, if

$$\lambda_p(A) \in \mathcal{E}_{\mathbf{i}}, \quad \lambda_q(A) \in \mathcal{E}_{\mathbf{e}},$$

then, by (3) and (4) we have

$$\lambda_p(T_1) \in \{\lambda \in C : \frac{a-b}{a+b} < |\lambda| < 1\},$$

$$\lambda_q(T_1) \in \{\lambda \in C : 1 < |\lambda|\},$$

$$\lambda_p(T_1^{-1}) \in \{\lambda \in C : 1 < |\lambda| < \sqrt{\frac{a+b}{a-b}}\ \},$$

$$\lambda_q(T_1^{-1}) \in \{\lambda \in C : 0 < |\lambda| < 1\}.$$

Therefore

$$\lambda_p(\widetilde{T}_1) \in \{\lambda \in C : \frac{a-b}{a+b} < |\lambda| < \sqrt{\frac{a-b}{a+b}} < 1\},$$

$$\lambda_q(\widetilde{T}_1) \in \{\lambda \in C : 0 < |\lambda| < \frac{a-b}{a+b}\ \}.$$

Hence, by (5), the matrix $\mathcal{L}$ has no eigenvalues on the unit circle $\omega_{\mathbf{o}} = \{\lambda \in C : |\lambda| = 1\}$.

By $\mathcal{P}_{\mathbf{i}}$ we denote the projection onto the maximal invariant subspace of the matrix $\mathcal{L}$ corresponding to the eigenvalues lying in

$$\omega_{\mathbf{i}} = \{\lambda \in C : |\lambda| < 1\},$$

and $\mathcal{L}\mathcal{P}_{\mathbf{i}} = \mathcal{P}_{\mathbf{i}}\mathcal{L}$. By $\mathcal{P}_{\mathbf{e}}$ we denote the projection onto the maximal invariant subspace of the matrix $\mathcal{L}$ corresponding to the eigenvalues lying in

$$\omega_{\mathbf{e}} = \{\lambda \in C : |\lambda| > 1\},$$

and $\mathcal{L}\mathcal{P}_{\mathbf{e}} = \mathcal{P}_{\mathbf{e}}\mathcal{L}$,

$$\mathcal{P}_{\mathbf{i}} + \mathcal{P}_{\mathbf{e}} = \mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

By analogy with Example 4 from [9], one can show that there exist inverse matrices

$$(\mathcal{I} - \mathcal{L}^k)^{-1} \tag{6}$$

for sufficiently large $k \gg 1$, and

$$\|(\mathcal{I} - \mathcal{L}^k)^{-1} - \mathcal{P}_{\mathbf{i}}\| \to 0, \quad k \to \infty. \tag{7}$$

Indeed, since

$$\|\mathcal{L}^k\mathcal{P}_{\mathbf{i}}\| \to 0, \quad \|\mathcal{L}^{-k}\mathcal{P}_{\mathbf{e}}\| \to 0, \quad k \to \infty,$$

then, by Theorem 1, there exist the inverse matrices (6) for $k \gg 1$, and the limit (7) is true.

Taking into account the definitions of the projections $P_{\mathbf{i}}$, $P_{\mathbf{e}}$, $\mathcal{P}_{\mathbf{i}}$ and $\mathcal{P}_{\mathbf{e}}$, from (5) it follows that

$$\mathcal{P}_{\mathbf{i}} = \mathcal{V}\begin{pmatrix} P_{\mathbf{i}} & 0 \\ 0 & I \end{pmatrix}\mathcal{V}^{-1}, \quad \mathcal{P}_{\mathbf{e}} = \mathcal{V}\begin{pmatrix} P_{\mathbf{e}} & 0 \\ 0 & 0 \end{pmatrix}\mathcal{V}^{-1}. \tag{8}$$

Let

$$\begin{pmatrix} \mathcal{N}_{11}^k & \mathcal{N}_{12}^k \\ \mathcal{N}_{21}^k & \mathcal{N}_{22}^k \end{pmatrix} = (\mathcal{I} - \mathcal{L}^k)^{-1}, \quad k \gg 1.$$

From (7), (8) we have

$$\| \begin{pmatrix} \mathcal{N}_{11}^k & \mathcal{N}_{12}^k \\ \mathcal{N}_{21}^k & \mathcal{N}_{22}^k \end{pmatrix} \mathcal{V} - \mathcal{V} \begin{pmatrix} P_{\mathbf{i}} & 0 \\ 0 & I \end{pmatrix} \| \to 0, \quad k \to \infty.$$

By (5), it follows that

$$\|\mathcal{N}_{11}^k + \mathcal{N}_{12}^k T_1 - P_{\mathbf{i}}\| \to 0, \quad \|\mathcal{N}_{11}^k + \mathcal{N}_{12}^k \widetilde{T}_1 - I\| \to 0,$$

$$\|\mathcal{N}_{21}^k + \mathcal{N}_{22}^k T_1 - P_{\mathbf{i}} T_1\| \to 0, \quad \|\mathcal{N}_{21}^k + \mathcal{N}_{22}^k \widetilde{T}_1 - \widetilde{T}_1\| \to 0, \quad k \to \infty.$$

By the first and the second limits, we obtain

$$\|\mathcal{N}_{12}^k (T_1 - \widetilde{T}_1) - P_{\mathbf{i}} + I\| \to 0, \quad k \to \infty.$$

The focuses of the ellipse $\mathcal{E}_{\mathbf{o}}$ are not eigenvalues of the matrix A, then there exists the inverse matrix $(T_1 - \widetilde{T}_1)^{-1}$. Hence,

$$\|\mathcal{N}_{12}^k - (P_{\mathbf{i}} - I)(T_1 - \widetilde{T}_1)^{-1}\| \to 0, \quad k \to \infty,$$

and we have

$$\|\mathcal{N}_{11}^k + (P_{\mathbf{i}} - I)(T_1 - \widetilde{T}_1)^{-1} \widetilde{T}_1 - I\| \to 0, \quad k \to \infty.$$

By analogy, from the third and the fourth limits it follows

$$\|\mathcal{N}_{22}^k (T_1 - \widetilde{T}_1) - P_{\mathbf{i}} T_1 + \widetilde{T}_1\| \to 0$$

or

$$\|\mathcal{N}_{22}^k - P_{\mathbf{i}} (T_1 - \widetilde{T}_1)^{-1} T_1 + (T_1 - \widetilde{T}_1)^{-1} \widetilde{T}_1\| \to 0, \quad k \to \infty.$$

Using these limits, we prove that the convergence

$$\|\mathcal{N}_{11}^k + \mathcal{N}_{22}^k - I - P_{\mathbf{i}}\| \to 0, \quad k \to \infty,$$

holds. Hence, we obtain the approximate projections for $P_{\mathbf{i}}$ and $P_{\mathbf{e}}$:

$$P_{\mathbf{i}} \approx \mathcal{N}_{11}^k + \mathcal{N}_{22}^k - I,$$

$$P_{\mathbf{e}} \approx 2I - \mathcal{N}_{11}^k - \mathcal{N}_{22}^k, \quad k \gg 1.$$

Remark 4. By analogy with Example 4 from [9], one can consider the case when at least one of the matrices

$$A - cI, \quad A + cI, \quad c^2 = a^2 - b^2,$$

is singular, i. e. at least one focus of the ellipse $\mathcal{E}_{\mathbf{o}}$ is an eigenvalue of the matrix A (fig. 2). To construct approximate projections one can use Theorems 1 and 2 (see [9]).

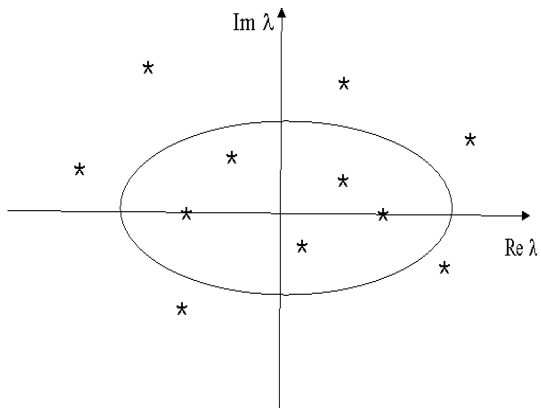

Fig. 2.

4. Lyapunov type equation in the elliptic dichotomy problem

In Section 3 we presented two variants of constructing approximate projections for $P_{\mathbf{i}}$, $P_{\mathbf{e}}$.

Recall that, by $P_{\mathbf{i}}$ we denoted the projection onto the maximal invariant subspace of the matrix A corresponding to the eigenvalues lying in $\mathcal{E}_{\mathbf{i}}$ and $AP_{\mathbf{i}} = P_{\mathbf{i}}A$. By $P_{\mathbf{e}}$ we denoted the projection onto the maximal invariant subspace of the matrix A corresponding to the eigenvalues lying in $\mathcal{E}_{\mathbf{e}}$ and $AP_{\mathbf{e}} = P_{\mathbf{e}}A$, $P_{\mathbf{i}} + P_{\mathbf{e}} = I$.

Problem. Our aim is to establish an algebraic criterion of the matrix spectrum dichotomy with respect to the ellipse $\mathcal{E}_{\mathbf{o}}$.

Consider the following matrix equation

$$(9)\quad H - (\frac{1}{2a^2} + \frac{1}{2b^2})A^*HA - (\frac{1}{4a^2} - \frac{1}{4b^2})(HA^2 + (A^*)^2H) = C.$$

Note, that in the case of $a = b = 1$ equation (9) is the discrete Lyapunov matrix equation

$$(10)\qquad H - A^*HA = C.$$

Using this matrix equation one can formulate a criterion for the matrix spectrum to belong to the unit disk $\{|\lambda| < 1\}$ (see, for example, [1]).

Recall the following classical result.

Theorem (Lyapunov). *All eigenvalues of the matrix A belong to the unit disk if and only if for a matrix $C = C^* > 0$ there exists a solution $H = H^* > 0$ of the matrix equation (10).*

Using the matrix equation (9), one can formulate an analogous criterion for the matrix spectrum to belong to $\mathcal{E}_{\mathbf{i}}$.

Theorem 3. *Let C be a Hermitian positive definite matrix. If there exists a solution $H = H^* > 0$ of the equation (9), then all eigenvalues of the matrix A belong to $\mathcal{E}_{\mathbf{i}}$.*

Proof. Suppose that there is an eigenvalue λ_m of A with

$$\frac{(\mathrm{Re}\,\lambda_m)^2}{a^2} + \frac{(\mathrm{Im}\,\lambda_m)^2}{b^2} \geq 1.$$

Then, by (9), for a corresponding eigenvector v_m we have

$$\langle Hv_m,\, v_m\rangle - (\frac{1}{2a^2} + \frac{1}{2b^2})\langle HAv_m,\, Av_m\rangle$$

$$-(\frac{1}{4a^2} - \frac{1}{4b^2})(\langle HA^2v_m,\, v_m\rangle + \langle Hv_m,\, A^2v_m\rangle) = \langle Cv_m,\, v_m\rangle$$

or

$$\langle Hv_m,\, v_m\rangle(1 - (\frac{1}{2a^2} + \frac{1}{2b^2})|\lambda_m|^2$$

$$-(\frac{1}{4a^2} - \frac{1}{4b^2})(\lambda_m^2 + \bar{\lambda}_m^2)) = \langle Cv_m,\, v_m\rangle.$$

Hence

$$\langle Hv_m,\, v_m\rangle(1 - \frac{(\mathrm{Re}\,\lambda_m)^2}{a^2} - \frac{(\mathrm{Im}\,\lambda_m)^2}{b^2}) = \langle Cv_m,\, v_m\rangle > 0.$$

However, $\langle Hv_m,\, v_m\rangle > 0$ and we have a contradiction, i.e., all eigenvalues of A belong to $\mathcal{E}_{\mathbf{i}}$. □

Theorem 4. *Let C be a Hermitian negative definite matrix. If there exists a solution $H = H^* > 0$ of the equation (9), then all eigenvalues of the matrix A belong to $\mathcal{E}_{\mathbf{e}}$.*

The proof of the theorem is analogous to the proof of Theorem 3.

Now we consider the matrix equation (9) with the special right-hand side:

$$H - (\frac{1}{2a^2} + \frac{1}{2b^2})A^*HA - (\frac{1}{4a^2} - \frac{1}{4b^2})(HA^2 + (A^*)^2H)$$

$$= P^*CP - (I-P)^*C(I-P), \quad P^2 = P. \tag{11}$$

Note that in the case of $a = b = 1$ the equation (11) has the form

$$H - A^*HA = P^*CP - (I-P)^*C(I-P). \tag{12}$$

This equation was studied by M. G. Krein (see [1]). Using the equation (12), H. Bulgak and S. K. Godunov elaborated an algorithm with guaranteed accuracy for solving the dichotomy problem for the matrix spectrum with respect to a circle (see, for example, [6]).

One can prove the following theorem.

Theorem 5. *Let a projection P commute with A: $AP = PA$. If for a matrix $C = C^* > 0$ there exists a solution $H = H^* > 0$ of the matrix equation (11) such that*

$$H = P^*HP + (I-P)^*H(I-P), \tag{13}$$

then the matrix A has no eigenvalues on the ellipse $\mathcal{E}_{\mathbf{o}}$. Moreover, P is the projection onto the maximal invariant subspace of the matrix A corresponding to the eigenvalues lying in $\mathcal{E}_{\mathbf{i}}$, i.e. $P = P_{\mathbf{i}}$.

Proof. From conditions on P the projection $(I - P)$ is one onto an invariant subspace of the matrix A. Hence, there exists a nonsingular matrix T such that

$$A = T\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} T^{-1}$$

and

$$P = T\begin{pmatrix} I_{11} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}.$$

Using the matrix T, we can rewrite the condition (13) in the form

$$T^*HT = T^*P^*T^{*-1}(T^*HT)T^{-1}PT$$

$$+T^*(I-P^*)T^{*-1}(T^*HT)T^{-1}(I-P)T.$$

Hence, for the matrix

$$\hat{H} = T^*HT = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

we have

$$\hat{H} = \begin{pmatrix} H_{11} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & H_{22} \end{pmatrix}.$$

Consequently,

$$\hat{H} = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}.$$

By $H = H^* > 0$, we have

$$H_{11} = H_{11}^* > 0, \quad H_{22} = H_{22}^* > 0.$$

By analogy, one can rewrite the matrix equation (11) as follows

$$\hat{H} - (\frac{1}{2a^2} + \frac{1}{2b^2})\hat{A}^*\hat{H}\hat{A} - (\frac{1}{4a^2} - \frac{1}{4b^2})(\hat{H}\hat{A}^2 + (\hat{A}^*)^2\hat{H})$$

$$= \hat{P}^*\hat{C}\hat{P} - (I - \hat{P})^*\hat{C}(I - \hat{P}),$$

where

$$\hat{A} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} I_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{C} = T^*CT = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

Therefore the matrix equation (11) is equivalent to the equations

$$H_{11} - (\frac{1}{2a^2} + \frac{1}{2b^2})A_{11}^*H_{11}A_{11} - (\frac{1}{4a^2} - \frac{1}{4b^2})(H_{11}A_{11}^2 + (A_{11}^*)^2H_{11}) = C_{11},$$

$$H_{22} - (\frac{1}{2a^2} + \frac{1}{2b^2})A_{22}^*H_{22}A_{22} - (\frac{1}{4a^2} - \frac{1}{4b^2})(H_{22}A_{22}^2 + (A_{22}^*)^2H_{22}) = -C_{22}.$$

The matrices

$$H_{11},\ H_{22},\ C_{11},\ C_{22}$$

are positive definite. From Theorems 3 and 4 it follows that eigenvalues of the matrix A_{11} belong to $\mathcal{E}_{\mathbf{i}}$ and eigenvalues of the matrix A_{22} belong to $\mathcal{E}_{\mathbf{e}}$. Obviously, $P = P_{\mathbf{i}}$. □

Remark 5. Note, the condition (13) provides the uniqueness of the solution of the matrix equation (11). This condition was introduced by M. G. Krein for the equation (12) (see, for example, [1, Ch. 1]).

5. Appendix

Using Theorems 1 and 2, one can obtain formulae for approximate projections in the dichotomy problems for the matrix spectrum with respect to a parabola or hyperbola. For these problems one can write matrix equations of Lyapunov type and prove corresponding theorems on location of the matrix spectrum with respect to a parabola or hyperbola. Using Theorems 1 and 2, one can obtain formulae for approximate projections in the trichotomy problems for the matrix spectrum with respect to an ellipse, parabola or hyperbola.

Acknowledgment. The author expresses his thanks to Dr. I. I. Matveeva for helpful discussions.

References

1. Daleckiĭ, Ju. L. and Krein M. G. (1974): *Stability of Solutions of Differential Equations in Banach Space*, Translations of Mathematical Monographs. 43. Providence, RI: American Mathematical Society.
2. Reed, M. and Simon, B. (1975): *Methods of Modern Mathematical Physics*, Vol. 1, Functional Analysis, Academic Press, New York.
3. Abramov, A. A. (1971): Boundary conditions at a singularity for systems of linear ordinary differential equations [in Russian], Zh. Vychisl. Mat. i Mat. Fiz. **11**, No. 1, 275–278.
4. Roberts, J. D. (1980): Linear model reduction and solution of the algebraic Riccati equation by use of the sign-function, Int. J. Control. **32**, No. 4, 677–687.
5. Bulgakov, A. Ya. (1989): The basis of guaranteed accuracy in the problem of separation of invariant subspaces for non-self-adjoint matrices [in Russian], Trudy Inst. Mat. **15**, 12–93; English transl. in Sib. Adv. Math. **1**, No. 1, 64–108 (1991) and Sib. Adv. Math. **1**, No. 2, 1–56 (1991).
6. Godunov, S. K. (1998): *Modern Aspects of Linear Algebra*, Translations of Mathematical Monographs. 175. Providence, RI: American Mathematical Society.
7. Demidenko, G. V. (1996): *On construction of approximate projections*, Novosibirsk, Russia, 10 p. (Preprint. Russian Acad. Sci., Siberian Branch. Sobolev Institute of Mathematics; N 38).
8. Demidenko, G. V. (1998): On a functional approach to constructing projections onto invariant subspaces of matrices [in Russian], Sibirsk. Mat. Zh. **39**, No. 4, 796–813; English transl. in Siberian Math. J. **39**, No. 4, 683–699.
9. Demidenko, G. V. (2001): On constructing approximate projections onto invariant subspaces of linear operators, Intern. J. Diff. Eq. Appl. **3**, No. 2, 135–146.
10. Godunov, S. K., Kiriluk, O. P. and Kostin, V. I. (1990): *Spectral portraits of matrices*, [in Russian], Novosibirsk, Russia, 10 p. (Preprint. Russian Acad. Sci., Siberian Branch. Institute of Mathematics; N 3).
11. Trefethen, L. N. (1991): Pseudospectra of matrices, in: Numerical Analysis, D. F. Griffiths and G. A. Watson, eds., Longman Scientific and Technical, Harlow, U.K., 1992, 234–266.
12. Bulgak, H. (1999): Pseudoeigenvalues, spectral portrait of a matrix and their connections with different criteria of stability, in: Error Control and Adaptivity in Scientific Computing, H. Bulgak and C. Zenger, eds., NATO Science Series, Kluwer Academic Publishers, 95–124.
13. Godunov, S. K. and Sadkane, M. (1996): Elliptic dichotomy of a matrix spectrum, Linear Algebra Appl. **248**, 205-232.
14. Lavrentiev, M. A. and Shabat, B. V. (1987): *Methods of the theory of functions of complex variable* [in Russian], Nauka, Moscow.