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Continuity of numeric characteristics for asymptotic stability of solutions to linear difference equations with periodic coefficients

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Summary. We consider linear perturbation systems of difference equations $y(n+1) = (A(n)+B(n))y(n)$, $n \geq 0$, where $A(n)$, $B(n)$ are $N \times N$ periodic matrices with period T. The spectrum of a monodromy matrix of the system $x(n+1) = A(n)x(n)$, $n \geq 0$, belongs to the unit disk $\{|\lambda| < 1\}$. We indicate conditions on a perturbation matrix $B(n)$ for asymptotic stability of the zero solution to the perturbation system and prove continuity one numeric characteristic of the asymptotic stability from [1].

Key words: monodromy matrix, perturbation systems of difference equations, asymptotic stability of solutions

Mathematics Subject Classification (1991): 39A11

1. Introduction

In this paper we consider linear perturbation systems of difference equations with periodic coefficients:

$$y(n+1) = (A(n) + B(n))y(n), \quad n \geq 0, \tag{1.1}$$

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where $A(n)$ and $B(n)$ are $N \times N$ periodic matrices with period T, i. e.

$$A(n+T) = A(n), \quad B(n+T) = B(n), \quad n \geq 0.$$

We suppose, that the zero solution of the system

$$x(n+1) = A(n)x(n), \quad n \geq 0, \tag{1.2}$$

is asymptotically stable. By the Spectral Criterion, each eigenvalue of the monodromy matrix of the system

$$X(T) = A(T-1)\dots A(1)A(0) \tag{1.3}$$

belongs to the unit disk $\{|\lambda| < 1\}$. According to the Lyapunov Criterion, the zero solution to system (1.2) is asymptotically stable if and only if the series

$$F = \sum_{k=0}^{\infty} (X^*(T))^k (X(T))^k \tag{1.4}$$

converges (see, for example, [2, 3]). Note that the matrix series F is a solution to the discrete matrix Lyapunov equation

$$X^*(T)FX(T) - F = -I. \tag{1.5}$$

In the paper [1] we indicated some numeric characteristics of the asymptotic stability of the zero solution to system (1.1) without appealing to the spectrum of the monodromy matrix (1.3). Using these characteristics one can obtain various estimates for solutions $\{x(n)\}$ to system (1.2). In particular, we considered the following numeric characteristic for asymptotic stability of the zero solution to (1.2)

$$\omega_1(A,T) = \|F\|, \tag{1.6}$$

where $\|F\|$ is the spectral norm of the matrix series (1.4).

The following statement holds [1].

Theorem 1. *The solution to system (1.2) satisfies the estimates*

$$\|x(n)\| \leq \|X(m)\| \left(1 - \frac{1}{\omega_1(A,T)}\right)^{k/2} \omega_1(A,T)^{1/2} \|x(0)\|,$$

$$n = kT + m, \quad k \geq 0, \quad 0 \leq m \leq T-1.$$

Our aim is to obtain conditions on a perturbation matrix $B(n)$ for asymptotic stability of the zero solution to system (1.1) and prove continuity of the norm of series (1.4).

2. Conditions for asymptotic stability of solutions

In this section we obtain conditions on a perturbation matrix $B(n)$ for asymptotic stability of the zero solution to system (1.1).

Let

$$Y(T) = (A(T-1) + B(T-1)) \dots (A(1) + B(1))(A(0) + B(0))$$

be the monodromy matrix of system (1.1).

Theorem 2. *If the perturbation matrix $B(n)$ such that the inequality holds*

$$\|Y(T) - X(T)\| < \sqrt{\|X(T)\|^2 + \frac{1}{\omega_1(A,T)}} - \|X(T)\|, \tag{2.1}$$

then the zero solution to system (1.1) is asymptotically stable.

Proof. Note that inequality (2.1) is equivalent to

$$\alpha = 1 - (2\|X(T)\|\|Y(T) - X(T)\| + \|Y(T) - X(T)\|^2)\|F\| > 0. \tag{2.2}$$

By T-periodicity, $Y(kT) = (Y(T))^k$, $k \geq 0$, therefore the sequence $\{Y(kT)\}$ is the solution to the problem

$$\begin{aligned} &Y((k+1)T) = (X(T) + (Y(T) - X(T))Y(kT), \quad k \geq 0, \\ &Y(0) = I. \end{aligned} \tag{2.3}$$

Consider the form

$$\langle FY((k+1)T)v, Y((k+1)T)v\rangle, \quad v \in E_N.$$

By (1.5) and (2.3), we have

$$\begin{aligned} &\langle FY((k+1)T)v, Y((k+1)T)v\rangle \\ &= \langle FY(kT)v, Y(kT)v\rangle - \langle Y(kT)v, Y(kT)v\rangle \\ &+\langle (X^*(T)F(Y(T) - X(T)) + (Y(T) - X(T))^*FX(T) \\ &+(Y(T) - X(T))^*F(Y(T) - X(T)))Y(kT)v, Y(kT)v\rangle. \end{aligned}$$

Using (2.2), we obtain the inequality

$$\langle FY((k+1)T)v, Y((k+1)T)v\rangle \leq \left(1 - \frac{\alpha}{\|F\|}\right) \langle FY(kT)v, Y(kT)v\rangle.$$

Consequently, for every $k \geq 0$ we have

$$\langle FY(kT)v, Y(kT)v\rangle \leq \left(1 - \frac{\alpha}{\|F\|}\right)^k \langle Fv, v\rangle.$$

The matrix F is positive definite. Hence, for every vector $v \in E_N$

$$\|(Y(T))^k v\| = \|Y(kT)v\| \to 0, \quad k \to \infty.$$

Then the spectrum of the monodromy matrix $Y(T)$ must lie in the disk $\{|\lambda| < 1\}$. According to the Spectral Criterion, the zero solution to system (1.1) is asymptotically stable. □

Corollary 1. *Let*

$$a = \max_{0 \le j \le T-1} \{\|A(j)\|\}, \quad b = \max_{0 \le j \le T-1} \{\|B(j)\|\}. \tag{2.4}$$

If the perturbation matrix $B(n)$ such that the inequality holds

$$Tb(a+b)^{T-1} < \sqrt{\|X(T)\|^2 + \frac{1}{\|F\|}} - \|X(T)\|, \tag{2.5}$$

then the zero solution to system (1.1) is asymptotically stable.

Proof. It is easy to verify that the inequality holds

$$\|Y(T) - X(T)\| \le (a+b)^T - a^T \le Tb(a+b)^{T-1}.$$

Therefore, asymptotic stability of the zero solution to system (1.1) is immediate from (2.1), (2.5). □

3. Continuity of numeric characteristics for asymptotic stability of solutions

We will suppose, that for the perturbation matrix $B(n)$ the condition of theorem 2 is true. Hence, the zero solution to system (1.1) is asymptotically stable. Then one can consider the matrix series

$$\widetilde{F} = \sum_{k=0}^{\infty} (Y^*(T))^k (Y(T))^k. \tag{3.1}$$

The matrix $\widetilde{F}$ is a solution to the discrete matrix Lyapunov equation

$$Y^*(T)\widetilde{F}Y(T) - \widetilde{F} = -I. \tag{3.2}$$

By analogy with (1.6), we define the numeric characteristic for asymptotic stability of the zero solution to (1.1) to be

$$\omega_1(A+B,T) = \|\widetilde{F}\|.$$

According to theorem 1, the estimates

$$\|y(n)\| \le \|Y(m)\| \left(1 - \frac{1}{\omega_1(A+B,T)}\right)^{k/2} \omega_1(A+B,T)^{1/2}\|y(0)\|,$$
$$n = kT + m, \quad k \ge 0, \quad 0 \le m \le T-1,$$

hold for solutions to system (1.1).

Theorem 3. *If the perturbation matrix $B(n)$ such that inequality (2.1) holds, then*

$$\|\widetilde{F} - F\| \le \frac{1-\alpha}{\alpha}\omega_1(A,T), \tag{3.3}$$

where the constant α is defined by (2.2).

Proof. By (1.5) and (3.2), we have

$$Y^*(T)(\widetilde{F} - F)Y(T) - (\widetilde{F} - F) = -C, \tag{3.4}$$

where

$$C = X^*(T)F(Y(T) - X(T)) + (Y(T) - X(T))^* F X(T)$$
$$+ (Y(T) - X(T))^* F(Y(T) - X(T)).$$

The discrete matrix Lyapunov equation (3.4) has a unique solution

$$\widetilde{F} - F = \sum_{k=0}^{\infty} (Y^*(T))^k C (Y(T))^k.$$

Hence, by (3.1), we have the inequality

$$\|\widetilde{F} - F\| \le \|C\|\|\widetilde{F}\| \le \|C\|\|\widetilde{F} - F\| + \|C\|\|F\|. \tag{3.5}$$

Obviously,

$$\|C\| \le (2\|X(T)\|\|Y(T) - X(T)\| + \|Y(T) - X(T)\|^2)\|F\| = 1 - \alpha,$$

and we obtain (3.3). □

Corollary 2. *If the perturbation matrix $B(n)$ such that the inequality holds*

$$\|Y(T) - X(T)\| < \sqrt{\|X(T)\|^2 + \frac{1}{2\omega_1(A,T)}} - \|X(T)\|, \tag{3.6}$$

then

$$\begin{aligned} \|\widetilde{F} - F\| &\le 2\|Y(T) - X(T)\|\omega_1(A,T) \\ &\times \left(\|X(T)\| + \sqrt{\|X(T)\|^2 + \tfrac{1}{2\omega_1(A,T)}}\right). \end{aligned} \tag{3.7}$$

Proof. By (3.5) and (3.6), one can obtain

$$\frac{1}{2}\|\widetilde{F}-F\| \leq (2\|X(T)\|+\|Y(T)-X(T)\|)\|Y(T)-X(T)\|\|F\|$$

$$\leq \left(\|X(T)\|+\sqrt{\|X(T)\|^2+\frac{1}{2\omega_1(A,T)}}\right)\|Y(T)-X(T)\|\|F\|.$$

Hence, we come to (3.7). □

Remark 1. Some analogous results for difference equations with constant coefficients are given in [4–6].

Remark 2. In the paper [7] we consider another numeric characteristics for asymptotic stability of solutions to (1.1) from [1].

Remark 3. A survey of results on numeric characteristics for asymptotic stability of solutions to linear differential equations with constant coefficients $\frac{dy}{dt}=Ay$ is given in [6]. Results for the case of linear differential equations with periodic coefficients $\frac{dy}{dt}=A(t)y$, $A(t+T)=A(t)$, were obtained in [8].

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