Selçuk J. Appl. Math.
Vol. 2, No. 1, pp. 95–106, 2001

Selçuk Journal of Applied Mathematics

Hierarchical cubature formulas

V. L. Vaskevich

Sobolev Institute of Mathematics, SB RAS, Novosibirsk, Russia;
e-mail: `vask@math.nsc.ru`

Received: March 10, 2001

Summary. We study the properties of hierarchical bases in the space of continuous functions with bounded domain and construct the hierarchical cubature formulas. Hierarchical systems of functions are similar to the well-known Faber – Schauder basis. It is shown that arbitrary hierarchical basis generates a scale of Hilbert subspaces in the space of continuous functions. The scale in many respects is similar to the usual classification of functional spaces with respect to smoothness. By integration over initial domain the standard interpolation formula for the given continuous integrand, we construct the hierarchical cubature formulas and prove that each of these formulas is optimal simultaneously in all Hilbert subspaces associated with the initial hierarchical basis. Hence, we have constructed the universally optimal cubature formulas.

Key words: calculation of integrals, guaranteed accuracy, cubature formulas, bases in Sobolev spaces, hierarchical bases

Mathematics Subject Classification (1991): 65D32, 65D30, 41A55

1. Introduction

Let Ω be a bounded domain in R^n; k is a non-negative integer; and $\Delta_k = \{x_j^{(k)} \in \overline{\Omega} \mid j = 1, 2, \ldots, N(k)\}$ is a finite subset of $\overline{\Omega}$. We assume that $\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_k \subset \cdots \subset \overline{\Omega}$ and the union of all Δ_k

is dense in $\overline{\Omega}$, i.e.,

$$\overline{\Big(\bigcup_{k=0}^{\infty} \Delta_k\Big)} = \overline{\Omega}.$$

The sequence $\boldsymbol{\Delta} = \{\Delta_k\}_{k=0}^{\infty}$ is said to be *a multigrid in* $\overline{\Omega}$; Δ_k is a k–level of $\boldsymbol{\Delta}$; and vectors $x_j^{(k)} \in \overline{\Omega}$ are nodes of $\boldsymbol{\Delta}$.

Given a positive integer k, we introduce two real numbers $\bar{h}_k$ and $\underline{h}_k$ by putting

$$(1.1) \qquad \bar{h}_k = \sup_{x\in\overline{\Omega}}\big\{\inf_{y\in\Delta_k} |x-y|\big\}; \quad \underline{h}_k = \inf_{x\neq y}\big\{|x-y| \mid x, y \in \Delta_k\big\}.$$

By the definition, $0 < \underline{h}_k \leq \bar{h}_k$. If $\bar{h}_k$ tends to zero as $k \to \infty$ then the union of Δ_k is obviously dense in $\overline{\Omega}$. We also assume that

$$(1.2) \quad \underline{h}_{k+1} < \underline{h}_k \text{ and } \exists\, \alpha \in (0,1): \ \alpha\overline{h}_k \leq \underline{h}_k \leq \overline{h}_k, \ \ k = 0, 1, 2, \ldots.$$

Together with k–levels of the multigrid $\boldsymbol{\Delta}$ we deal with their differences defined as follows

$$\Delta_k \setminus \Delta_{k-1} = \{\tilde{x}_j^{(k)} \mid j = 1, 2, \ldots, \sigma(k)\}, \quad k = 1, 2, \ldots.$$

If $k = 0$ then $\tilde{x}_j^{(0)} = x_j^{(0)}$, $j = 1, 2, \ldots, \sigma(0) = N(0)$. It is obvious that $\sigma(0) + \sigma(1) + \cdots + \sigma(k) = N(k)$.

Let $C(\overline{\Omega})$ be the Banach space of functions which are continuous in $\overline{\Omega}$. For $u \in C(\overline{\Omega})$ and $x \in \overline{\Omega}$ the value of u at $x \in \overline{\Omega}$ is defined. Hence, we can consider the vectors

$$A_k(\Delta \mid u) = \big(u(\tilde{x}_1^{(k)}), u(\tilde{x}_2^{(k)}), \ldots, u(\tilde{x}_{\sigma(k)}^{(k)})\big),$$

where $k = 0, 1, \ldots$. From the vectors $A_k(\Delta \mid u)$ we compose the following infinite sequence

$$u_\Delta = (A_1(\Delta \mid u), A_2(\Delta \mid u), \ldots, A_k(\Delta \mid u), \ldots).$$

We will operate by u_Δ as an infinite column-vector.

By the definition, we have

$$||u_\Delta \mid l_\infty|| = \sup_{x\in\boldsymbol{\Delta}} |u(x)| \leq \sup_{x\in\overline{\Omega}} |u(x)| = ||u \mid C(\overline{\Omega})||.$$

Whence the linear operator $T_\Delta : C(\overline{\Omega}) \to l_\infty$ transformed the functions $u(x)$ from $C(\overline{\Omega})$ into the sequence $u_\Delta \in l_\infty$ is bounded. We call the vector $u_\Delta = T_\Delta(u)$ *a trace of u on the multigrid* $\boldsymbol{\Delta}$.

Let H be a countable subset of $C(\overline{\Omega})$,

$$H = \{h_j^{(k)}(x) \in C(\overline{\Omega}) \mid j = 1, 2, \ldots, \sigma(k);\ k = 0, 1, \ldots\}.$$

In this paper, we consider H such that for members of H the following equalities hold

$$\begin{array}{lll} h_j^{(k)}(\tilde{x}_i^{(m)}) = 0,\ \forall m < k; & k = 0, 1, 2, \ldots, \\ h_j^{(k)}(\tilde{x}_l^{(k)}) = \delta_j^l; & j, l = 1, 2, \ldots, \sigma(k). \end{array} \tag{1.3}$$

Here δ_j^l is the conventional Kronecker delta.

Definition 1. *If (1.3) holds then H is called a hierarchical system.*

From (1.3) it follows that all functions $h_1, h_2, \ldots, h_M$ of a hierarchical system H are linearly independent.

By the definition, if H is a hierarchical system and $h_j^{(k)}(x) \in H$ then the trace $\left(h_j^{(k)}\right)_{\boldsymbol{\Delta}}$ of $h_j^{(k)}$ on the multigrid $\boldsymbol{\Delta}$ has zeroes as the entries in positions $1, 2, \ldots, N(k-1)+j-1, N(k-1)+j+1, \ldots, N(k)$; and the entry of $\left(h_j^{(k)}\right)_{\boldsymbol{\Delta}}$ in position $N(k-1)+j$ equals 1.

Let H be a hierarchical system and let $U_{\boldsymbol{\Delta}}$ be the matrix which has the traces $\left(h_j^{(k)}\right)_{\boldsymbol{\Delta}}$ as columns, i.e.,

$$U_{\boldsymbol{\Delta}} = (h_{1,\boldsymbol{\Delta}}^{(0)}, h_{2,\boldsymbol{\Delta}}^{(0)}, \ldots, h_{\sigma(0),\boldsymbol{\Delta}}^{(0)}, \ldots, h_{1,\boldsymbol{\Delta}}^{(k)}, \ldots, h_{j,\boldsymbol{\Delta}}^{(k)}, \ldots, h_{\sigma(k),\boldsymbol{\Delta}}^{(k)}, \ldots).$$

Then $U_{\boldsymbol{\Delta}}$ is a subdiagonal matrix with 1 on the main diagonal.

By the same way as in definition 1, we can define a hierarchical system in a Hilbert space. In [8], [9], and [3], it was shown that hierarchical systems in Sobolev–like spaces may be constructed as sequences of interpolating D^m–splines. It should be noted that hierarchical systems are frequently applied to the solution of boundary value problems by the method of finite elements (see, e.g., [2], [5], [6], and [11]). Hierarchical systems in Hilbert spaces such as Sobolev–like spaces are studied in [4].

Definition 2. *If a Δ-hierarchical system H in $C(\overline{\Omega})$ is a basis of $C(\overline{\Omega})$, then H is called a hierarchical basis.*

Example 1. The well-known Faber — Schauder system is a hierarchical basis of $C[0,1]$, (see [7, p. 227]).

Given a non-negative integer m and a hierarchical basis

$$H = \{h_j^{(k)}(x) \in C(\overline{\Omega}) \mid j = 1, 2, \dots, \sigma(k);\ k = 0, 1, \dots\}$$

in $C(\overline{\Omega})$, we define the members of the following finite set

$$\{\omega_{l,m}^{(i)}(x) \mid i = 0, 1, \dots, m,\ l = 1, 2, \dots, \sigma(i)\}.$$

by the equalities

$$\begin{aligned} &\omega_{j,m}^{(m)}(x) = h_j^{(m)}(x), \quad j = 1, 2, \dots, \sigma(m),\\ &\omega_{l,m}^{(i)}(x) = h_l^{(i)}(x) - \sum_{k=i+1}^{m} \sum_{j=1}^{\sigma(k)} h_l^{(i)}(\tilde{x}_j^{(k)})\omega_{j,m}^{(k)}(x),\\ &\qquad i = m-1, m-2, \dots, 0, \quad l = 1, 2, \dots, \sigma(i). \end{aligned}$$

By the definition of H, the following equalities hold

$$\begin{aligned} &\omega_{l,m}^{(i)}(\tilde{x}_j^{(k)}) = \delta_i^k \delta_l^j, \quad i, k = 0, 1, \dots, m,\\ &l = 1, 2, \dots, \sigma(i), \quad j = 1, 2, \dots, \sigma(k). \end{aligned}$$

Given a continuous function $\varphi(x)$ with domain Ω, we consider the following interpolation formula

$$\varphi(x) \cong \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} \varphi(\tilde{x}_j^{(k)})\omega_{j,m}^{(k)}(x), \quad x \in \Omega.$$

By integration of the both sides of this approximate equality, we obtain the cubature formula

$$\int_{\Omega} \varphi(x)\, dx \cong \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} c_{j,m}^{(k),o} \varphi(\tilde{x}_j^{(k)}), \tag{1.4}$$

with $c_{j,m}^{(k),o}$ the weights defined as follows

$$c_{j,m}^{(k),o} = \int_{\Omega} \omega_{j,m}^{(k)}(x)\, dx, \quad k = 0, 1, \dots, m,\ j = 1, 2, \dots, \sigma(k). \tag{1.5}$$

The formula (1.4) will be referred to as *the hierarchical cubature formula*. Our goal in this paper is to study the properties of cubature formulas of the form (1.4).

2. Hilbert spaces associated with a hierarchical basis

Let $H = \{h_j^{(k)}(x) \mid j = 1, 2, \dots, \sigma(k);\ k = 0, 1, \dots\}$ be a hierarchical basis in $C(\overline{\Omega})$. Given a function $\varphi(x)$ from $C(\overline{\Omega})$, we can expand it in the series

$$\varphi(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(k)} g_j^{(k)} h_j^{(k)}(x). \tag{2.1}$$

The coefficients $g_j^{(k)} = g_j^{(k)}(\varphi)$ of (2.1) are uniquely determined by a function φ. The partial sums of (2.1) converge to $\varphi(x)$ in the norm of $C(\overline{\Omega})$. If $\varphi(x) = h_l^{(m)}(x)$ then we have

$$\begin{aligned} g_j^{(k)}(h_l^{(m)}) = \delta_l^j \delta_m^k, &\quad j = 1, \dots, \sigma(k), \\ l = 1, \dots, \sigma(m), &\quad k, m = 0, 1, 2, \dots. \end{aligned} \tag{2.2}$$

Together with the hierarchical basis H we introduce into consideration a numerical sequence $\{h(m)\}_{m=0}^{\infty}$, by putting

$$h(m) = \sup_{x \in \overline{\Omega}} \{|h_j^{(m)}(x)| \mid j = 1, 2, \dots, \sigma(m)\}.$$

Let $\{a_m\}_{m=0}^{\infty}$ be a sequence of positive numbers such that

$$L_H(a_m) \equiv \left\{ \sum_{m=0}^{\infty} \frac{\sigma(m) h^2(m)}{a_m^2} \right\}^{1/2} < \infty. \tag{2.3}$$

Given a sequence $\{a_m\}_{m=0}^{\infty}$, we define the linear subspace $X^{(a_m)}(\Omega)$ of $C(\overline{\Omega})$ as follows

$$X^{(a_m)}(\Omega) = \left\{ \varphi \in C(\overline{\Omega}) \mid \langle \varphi \rangle^2 = \sum_{m=0}^{\infty} a_m^2 \sum_{j=1}^{\sigma(m)} |g_j^{(m)}(\varphi)|^2 < \infty \right\}. \tag{2.4}$$

Since (2.2) holds, it follows that $h_l^{(k)}(x)$ belonging to H is also a member of $X^{(a_m)}(\Omega)$. Hence, $X^{(a_m)}(\Omega)$ is an infinite-dimensional linear space.

We introduce a bilinear form in the space $X^{(a_m)}(\Omega)$, by letting for all functions φ and ψ

$$\langle \varphi, \psi \rangle = \sum_{k=0}^{\infty} a_k^2 \sum_{j=1}^{\sigma(k)} g_j^{(k)}(\varphi) g_j^{(k)}(\psi). \tag{2.5}$$

Applying the Cauchy inequality for sums to the right side of (2.5), we obtain

$$|\langle \varphi, \psi \rangle| \leq \langle \varphi \rangle \langle \psi \rangle < \infty.$$

Hence, for φ and ψ from $X^{(a_m)}(\Omega)$ the series in the right side of (2.5) converges absolutely.

Theorem 1. *Functions from the hierarchical basis H are mutually orthogonal in the inner product $\langle \cdot, \cdot \rangle$ defined by (2.5). If the corresponding norm is denoted by $\langle \cdot \rangle$ then for a function φ belonging to $X^{(a_m)}(\Omega)$ (2.1) converges not only in the norm of $C(\overline{\Omega})$ but in the norm $\langle \cdot \rangle$ too.*

Proof. To begin with, we consider the bilinear form (2.5). By the definition, it follows that $\langle \varphi, \psi \rangle$ is a symmetric and linear form with respect to φ and ψ. By (2.5), for a function φ in $X^{(a_m)}(\Omega)$ the inequality $\langle \varphi, \varphi \rangle \geq 0$ holds. If $\varphi = 0$ then $g_j^{(k)}(\varphi) = 0$, and $\langle \varphi, \varphi \rangle = 0$. Conversely, let $\langle \varphi, \varphi \rangle = 0$ for a function φ in $X^{(a_m)}(\Omega)$. Then $g_j^{(k)}(\varphi) = 0$ for all k and j; and the partial sums of (2.1) are equal to 0 everywhere in domain Ω. Hence, the limit of these partial sums is also equal to 0, i.e., $\varphi = 0$.

Thus, we have proved that the bilinear form (2.5) is an inner product in $X^{(a_m)}(\Omega)$. The corresponding norm is defined by the equality $\langle \varphi \rangle = \langle \varphi, \varphi \rangle^{1/2}$.

Since (2.2) and (2.5) hold, it follows that

$$\begin{aligned}\langle h_l^{(m)}, h_j^{(k)} \rangle &= \sum_{i=0}^{\infty} a_i^2 \sum_{n=1}^{\sigma(i)} g_n^{(i)}(h_l^{(m)}) g_n^{(i)}(h_j^{(k)}) \\ &= \sum_{i=0}^{\infty} a_i^2 \sum_{n=1}^{\sigma(i)} \delta_l^n \delta_m^i \delta_j^n \delta_k^i = a_m^2 \delta_m^k \delta_j^l.\end{aligned}$$

Hence, functions from the hierarchical basis H are mutually orthogonal in the inner product $\langle \cdot , \cdot \rangle$, and for the norm of $h_l^{(m)}(x)$ in this inner product we have $\langle h_l^{(m)} \rangle = a_m$.

Let $\varphi_m(x)$ be the partial sum of (2.1), i.e.,

$$\varphi_m(x) = \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} g_j^{(k)} h_j^{(k)}(x).$$

The orthogonality of the basis functions $h_j^{(k)}(x)$ implies

$$\langle \varphi - \varphi_m \rangle^2 = \sum_{k=m+1}^{\infty} a_k^2 \sum_{j=1}^{\sigma(k)} |g_j^{(k)}(\varphi)|^2.$$

By the definition of $X^{(a_m)}(\Omega)$, the sequence of the sums in the right side of the last equality converges to 0 as m tends to infinity. It means that (2.1) converges to φ in the norm $\langle\cdot\rangle$. $\square$

Let H be a hierarchical basis in $C(\overline{\Omega})$ and let $\varphi_m(x)$ be the partial sum of the series (2.1), i.e.,

$$
\begin{aligned}
\varphi_0(x) &= \sum_{j=1}^{\sigma(0)} g_j^{(0)}(\varphi)h_j^{(0)}(x),\\
\varphi_m(x) &= \varphi_{m-1}(x) + \sum_{j=1}^{\sigma(m)} g_j^{(m)}(\varphi)h_j^{(m)}(x) \quad \text{for } m \geq 1.
\end{aligned} \tag{2.6}
$$

Then the following equalities hold

$$
\varphi_m(x_j^{(m)}) = \varphi(x_j^{(m)}), \quad j = 1, 2, \ldots, N(m). \tag{2.7}
$$

The function $\varphi_m(x)$ is said to be a standard interpolant for $\varphi(x)$.

Lemma 1. *For an arbitrary $\varphi(x) \in C(\overline{\Omega})$ coefficients $g_j^{(k)}(\varphi)$ of (2.1) may be defined by*

$$
\begin{aligned}
g_j^{(0)}(\varphi) &= \varphi(\tilde{x}_j^{(0)}),\ j = 1, 2, \ldots, \sigma(0);\\
g_j^{(m)}(\varphi) &= \varphi(\tilde{x}_j^{(m)}) - \varphi_{m-1}(\tilde{x}_j^{(m)}),\ j = 1, 2, \ldots, \sigma(m),\ m \geq 1.
\end{aligned} \tag{2.8}
$$

For a given positive integer N there is a positive real number A_N with

$$
\{\sum_{m=0}^{N} a_m^2 \sum_{j=1}^{\sigma(m)} |g_j^{(m)}(\varphi)|^2\}^{1/2} \leq A_N ||\varphi \mid C(\overline{\Omega})||. \tag{2.9}
$$

Here A_N does not depend on φ. Hence, linear functionals $g_j^{(k)}(\cdot)$ are bounded on $C(\overline{\Omega})$.

Proof. By the definition of H, (2.6), together with (2.7), yields (2.8). It is not hard to show by induction on m that the inequality holds

$$
\sup_{x\in\overline{\Omega}} |\varphi_m(x)| \leq G(m) \sup_{x\in\Delta_m} |\varphi(x)|,
$$

where $G(0) = \sigma(0)$ and $G(m) = \sigma(m)h(m) + G(m-1)(1+\sigma(m)h(m))$ for $m \geq 1$. This, together with (2.8), yields

$$
|g_j^{(m)}(\varphi)| \leq (1 + G(m-1)) \sup_{x\in\overline{\Omega}} |\varphi(x)|.
$$

Hence, (2.9) holds with $A_N = \{\sum_{m=0}^{N} \sigma(m)a_m^2 |1 + G(m-1)|^2\}^{1/2}$. $\square$

Theorem 2. *The space $X^{(a_m)}(\Omega)$ with the norm $\langle\cdot\rangle$ is complete and so it is a Hilbert space. The embedding of $X^{(a_m)}(\Omega)$ in $C(\overline{\Omega})$ is bounded and for an arbitrary function φ from $X^{(a_m)}(\Omega)$ the following inequality holds*

$$(2.10)\qquad ||\varphi \mid C(\overline{\Omega})|| \le L_H(a_m)\langle\varphi\rangle,$$

with $L_H(a_m)$ the constant defined by (2.3).

Proof. Let $\varphi(x)$ be a member of $X^{(a_m)}(\Omega)$. Then the corresponding series (2.1) converges to $\varphi(x)$ both in the norm of $C(\overline{\Omega})$ and in the norm of $X^{(a_m)}(\Omega)$. Whence and from the definition of $\{h(m)\}_{m=0}^{\infty}$ it follows that

$$|\varphi(x)| \le \sum_{k=0}^{\infty}\sum_{j=1}^{\sigma(k)} |g_j^{(k)}(\varphi)||h_j^{(k)}(x)| \le \sum_{k=0}^{\infty} h(k)\sum_{j=1}^{\sigma(k)} |g_j^{(k)}(\varphi)|.$$

Applying the Cauchy inequality for sums to the right side of this inequality, we obtain

$$|\varphi(x)| \le \sum_{k=0}^{\infty} h(k)\sigma^{1/2}(k)\{\sum_{j=1}^{\sigma(k)} |g_j^{(k)}(\varphi)|^2\}^{1/2} \le L_H(a_m)\langle\varphi\rangle.$$

Thus, we have proved (2.10).

Let $\{\varphi^{(k)}(x)\}_{k=1}^{\infty}$ be a Cauchy sequence in the space $X^{(a_m)}(\Omega)$. By (2.10), it also is a Cauchy sequence in the space $C(\overline{\Omega})$. Consequently, there exists a function $\varphi(x)$ in $C(\overline{\Omega})$ with

$$(2.11)\qquad \lim_{k\to\infty}\sup_{x\in\overline{\Omega}} |\varphi(x) - \varphi^{(k)}(x)| = 0.$$

Moreover, there exists a positive real number R such that

$$\sup_{k\ge 1}\langle\varphi^{(k)}\rangle \le R < \infty.$$

Let N be an integer and let $\varphi_N(x)$ be the partial sum of (2.1). Then

$$(2.12)\qquad \langle\varphi_N\rangle \le \{\sum_{m=0}^{N} a_m^2 \sum_{j=1}^{\sigma(m)} |g_j^{(m)}(\varphi - \varphi^{(k)})|^2\}^{1/2} + \sup_{k\ge 1}\langle\varphi^{(k)}\rangle.$$

To estimate the first summand in the right side of (2.12) we use (2.9) and obtain

$$(2.13)\qquad \langle\varphi_N\rangle \le A_N||\varphi - \varphi^{(k)} \mid C(\overline{\Omega})|| + \sup_{k\ge 1}\langle\varphi^{(k)}\rangle,$$

where A_N is independent of φ and $\varphi^{(k)}$. Since (2.11) and (2.13) hold it follows that for a given integer N there exists a number $K = K(N)$ such that for $k \geq K(N)$ the first summand in the right side of (2.13) is less then R. Hence,

$$\langle \varphi_N \rangle \leq 2R \quad \text{and} \quad \langle \varphi \rangle = \lim_{N \to \infty} \langle \varphi_N \rangle \leq 2R < \infty,$$

i.e., $\varphi(x)$ is a member of $X^{(a_m)}(\Omega)$.

Let us check that $\langle \varphi - \varphi^{(k)} \rangle \to 0$ as $k \to \infty$. It is well-known that there exists a completion $\overline{X}(\Omega)$ of a space $X^{(a_m)}(\Omega)$ with the inner product $\langle \cdot, \cdot \rangle$. In addition, $X^{(a_m)}(\Omega)$ is dense in $\overline{X}(\Omega)$. Since $\overline{X}(\Omega)$ is complete it follows that there exists an element $\psi \in \overline{X}(\Omega)$ such that $\langle \psi - \varphi^{(k)} \rangle \to 0$ as $k \to \infty$. By (2.10), we may consider ψ as a continuous function with domain Ω. Moreover, the initial sequence $\{\varphi^{(k)}(x)\}_{k=1}^{\infty}$ converges to ψ in the norm of $C(\overline{\Omega})$. By uniqueness of the limit, we conclude that $\varphi = \psi$. In other words, $X^{(a_m)}(\Omega)$ coincides with $\overline{X}(\Omega)$, i.e., $X^{(a_m)}(\Omega)$ is a Hilbert space. $\square$

Corollary 1. *Let $\varphi(x)$ be a member of $X^{(a_m)}(\Omega)$ and let $\varphi_m(x)$ be the partial sum of (2.1). Then*

$$\varphi(x) = \varphi_0(x) + \sum_{k=1}^{\infty} (\varphi_k(x) - \varphi_{k-1}(x)) = \sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(k)} g_j^{(k)}(\varphi) h_j^{(k)}(x), \tag{2.14}$$

The series in the right side of (2.14) converges to φ in the norm $\langle \cdot \rangle$.

By (2.14), we can split the identical operator into the direct sum of projections of $X^{(a_m)}(\Omega)$ to the finite-dimensional subspaces of $X^{(a_m)}(\Omega)$. Hence, (2.14) is similar to the multi-level splitting of finite element spaces (see [11, p. 383]).

3. The optimality of the hierarchical cubature formulas

In this section, we consider cubature formulas of the form

$$\int_{\Omega} \varphi(x)\, dx \cong \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} c_{j,m}^{(k)} \varphi(\tilde{x}_j^{(k)}). \tag{3.1}$$

Integrable functions are assumed to be members of some Hilbert space $X^{(a_k)}(\Omega)$ embedded into $C(\overline{\Omega})$; the nodes of formula (3.1) are the members of m-level Δ_m of the multigrid $\boldsymbol{\Delta}$; and the number of the nodes equals $N(m)$.

To each cubature formula (3.1) we assign *the error*

$$(3.2) \qquad (l_m, \varphi) = \int\limits_{\Omega} \varphi(x)\, dx - \sum_{k=0}^{m} \sum_{j=1}^{\sigma(k)} c_{j,m}^{(k)} \varphi(\tilde{x}_j^{(k)}).$$

The error is a linear functional, therefore also referred to as *error functional*, since we require that the rules for choosing the nodes and the weights of (3.1) be independent of specifying an integrable function. We consider the following problem.

Problem 1. *Given a Hilbert space $X^{(a_k)}(\Omega)$ and a positive integer m, find the error (3.2) with $N(m)$ nodes and the minimal norm in the space duel to $X^{(a_k)}(\Omega)$.*

The cubature formula corresponding to the solution of Problem1 is said to be an $X^{(a_k)}(\Omega)$-*optimal formula.*

Theorem 3. *For a positive integer m there is a unique $X^{(a_k)}(\Omega)$-optimal cubature formula of the form (3.1). The weights of $X^{(a_k)}(\Omega)$-optimal cubature formula are defined by (1.5), i.e., this formula is hierarchical cubature formula (1.4).*

Proof. By the Riesz Theorem, the error functional l_m defined by (3.2) may be written as inner product

$$(3.3) \qquad (l_m, \varphi) = \langle u_m, \varphi \rangle, \quad \forall \varphi \in X^{(a_k)}(\Omega).$$

Here u_m is a uniquely determined member of $X^{(a_k)}(\Omega)$ called the *extremal function* of l_m or, more verbosely, $X^{(a_k)}(\Omega)$-*extremal function.* Moreover, the following equalities hold

$$(3.4) \qquad ||l_m \mid X^{(a_k),*}(\Omega)||^2 = \langle u_m \rangle^2 = \sum_{k=0}^{\infty} a_k^2 \sum_{j=1}^{\sigma(k)} |g_j^{(k)}(u_m)|^2.$$

The extremal function u_m expands into a series in $h_j^{(k)}(x)$. Moreover, the coefficient $g_j^{(k)}(u_m)$ of $h_j^{(k)}(x)$ is defined by

$$(3.5) \qquad g_j^{(k)}(u_m) = \langle u_m, h_j^{(k)} \rangle / a_k^2 = (l_m, h_j^{(k)})/a_k^2.$$

This, together with (3.4), yields

$$(3.6) \qquad ||l_m \mid X^{(a_k),*}(\Omega)||^2 = \sum_{k=0}^{\infty} \sum_{j=1}^{\sigma(k)} \frac{|(l_m, h_j^{(k)})|^2}{a_k^2}.$$

If $k > m$ then the values of $h_j^{(k)}$ at the points of Δ_m are equal to 0. Hence, we get

$$(l_m, h_j^{(k)}) = \int\limits_{\Omega} h_j^{(k)}(x)\,dx \equiv b_j^{(k)} \quad \text{for} \quad k > m.$$

Inserting these equalities in (3.6), we obtain

$$(3.7) \qquad ||l_m \mid X^{(a_k),*}(\Omega)||^2 \geq \sum_{k=m+1}^{\infty} \sum_{j=1}^{\sigma(k)} \frac{|b_j^{(k)}|^2}{a_k^2}.$$

Let l_m^0 be the error corresponding to hierarchical cubature formula (1.4) and let u_m^0 be the extremal function of l_m^0. By the definition, we have

$$(l_m^0, h_j^{(k)}) = 0 \quad \text{for} \quad k = 0, 1, \ldots, m,\ j = 1, 2, \ldots, \sigma(k).$$

Whence and from (3.6) it follows that

$$(3.8) \qquad ||l_m^0 \mid X^{(a_k),*}(\Omega)||^2 = \sum_{k=m+1}^{\infty} \sum_{j=1}^{\sigma(k)} \frac{|b_j^{(k)}|^2}{a_k^2}.$$

Since (3.7) and (3.8) hold, the $X^{(a_k)}(\Omega)$-optimality of hierarchical cubature formula (1.4) is immediate.

By the parallelogram law, an $X^{(a_k)}(\Omega)$-optimal cubature formula is unique. □

It is well-known that the same cubature formula may be optimal simultaneously in many normed spaces not necessarily equivalent to one another. The formulas with such properties is called to be *universally optimal* cubature formulas (see, e.g., [1] and [10]). By Theorem 3, hierarchical cubature formula (1.4) is the universally optimal cubature formula on the family of Hilbert spaces introduced in section 2.

References

1. Babuška, I. (1968): Über universal optimale Quadratur Formeln, Apl. Mat. **13**, No. 4, 304–338 and No. 5, 388–404.
2. Bank, R.E., Dupont, T., and Yserentant, H. (1988): The hierarchical basis multigrid method, Numer. Math. **52**, 427–458.
3. Bezhaev, A.Yu. and Vasilenko, V.A. (1993): *Variational Spline Theory*, Bull. of Novosibirsk Computing Center, Series: Numerical Analysis, Special Issue: **3**.

4. Bulgak, H. and Vaskevich, V.L. (1999): Hierarchical bases in Hilbert spaces [in Russian], Siberian J. of Industrial Math. **2**(2), 24–35.
5. Bungartz, H.-J. (1997): A multigrid algorithm for higher order finite elements on sparse grids, Electronic Transaction on Numerical Analysis **6**, 63–77.
6. Bungartz, H.-J. and Zenger, C. (1999): Error control for adaptive sparse grids, in: Error Control and Adaptivity in Scientific Computing, Bulgak, H. and Zenger, C., (Eds.), Kluwer Academic Publishers.
7. Kashin, B.S. and Saakyan, A.A. (1984): *Orthogonal Bases*, Nauka, Moscow. English transl.: Transl. of Math. Monographs **75**, Amer. Math. Soc., Providence (1989).
8. Matveev, O.V. (1992): Spline interpolation of functions of several variables, and bases in Sobolev spaces, Tr. Mat. Inst. Steklova **198**, 125–152. English transl. in Proc. Steklov Inst. Math. **198**, 119–146 (1994).
9. Rozhenko, A.I. (1999): *The Abstract Theory of Splines* [in Russian], Novosibirsk University Press, Novosibirsk.
10. Sobolev, S. L. and Vaskevich, V. L. (1997): *The Theory of Cubature Formulas*, Kluwer Academic Publishers, Dordrecht.
11. Yserentant H. (1986): On the multilevel splitting of finite element spaces, Numer. Math. **49**, 379–412.