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Solvability conditions of the Cauchy problem for the system of internal waves

Inessa Matveeva[*]

Sobolev Institute of Mathematics, SB RAS, 630090 Novosibirsk, Russia;
e-mail: matveeva@math.nsc.ru

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Summary. The author studies the Cauchy problem for a system of integrodifferential equations describing internal waves for the Boussinesq approximation. The necessary and sufficient conditions of solvability in weighted Sobolev spaces are established.

Key words: integrodifferential equations, Cauchy problem, conditions of solvability, weighted Sobolev spaces

Mathematics Subject Classification (1991): 35M05, 35Q35

1. Introduction

In the present paper we study the Cauchy problem for the following system of integrodifferential equations

$$
\begin{aligned}
&D_t u_1 + D_{x_1} p = 0, \\
&D_t u_2 + D_{x_2} p = 0, \\
&D_t u_3 + \omega^2 \int_0^t u_3 \, ds + D_{x_3} p = 0, \\
&\operatorname{div} u = 0,
\end{aligned} \tag{1}
$$

where $\omega > 0$ is a real constant. The system describes internal waves for the Boussinesq approximation.

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System (1) is not a system of the Cauchy – Kovalevsky type. At present there are a great number of works devoted to the research of equations and systems not solved with respect to the highest derivative (for example, see the bibliography in the book [1]). These investigations show that the theory of boundary value problems for such equations and systems has certain peculiarities in comparison with the theory of boundary value problems for classical equations and systems.

2. Definitions and main results

Consider the Cauchy problem for (1) in the half-space $R_4^+ = \{(t,x) : t > 0,\ x \in R_3\}$

$$
(2)\qquad
\begin{aligned}
&D_t u_1 + D_{x_1} p = 0, && t > 0,\ x \in R_3,\\
&D_t u_2 + D_{x_2} p = 0, &&\\
&D_t u_3 + \omega^2 \int_0^t u_3\, ds + D_{x_3} p = 0, &&\\
&\operatorname{div} u = 0, &&\\
&u_j|_{t=0} = u_j^0(x),\ j = 1,2,3, && x \in R_3.
\end{aligned}
$$

Suppose that r_1 and r_2 are integers, $1 < q < \infty$, $\gamma > 0$. Introduce the following weighted functional spaces.

By $L_{q,\gamma}$ we denote the space of functions $v(t,x)$ with the norm

$$\|v(t,x),\, L_{q,\gamma}(R_4^+)\| = \|e^{-\gamma t} v(t,x),\, L_q(R_4^+)\|.$$

By $W_{q,\gamma}^{r_1,r_2}(R_4^+)$ we denote the weighted Sobolev space of functions $v(t,x)$ with the norm

$$\|v(t,x),\, W_{q,\gamma}^{r_1,r_2}(R_4^+)\| = \|D_t^{r_1} v(t,x),\, L_{q,\gamma}(R_4^+)\|$$

$$+ \sum_{|\alpha| \le r_2} \|D_x^\alpha v(t,x),\, L_{q,\gamma}(R_4^+)\|.$$

By $L_{1,\sigma}(R_3)$ we denote the space of functions $v(x)$ with the norm

$$\|v(x),\, L_{1,\sigma}(R_3)\| = \|(1+|x|)^{-\sigma} v(x),\, L_1(R_3)\|.$$

Results obtained in [2] give the following assertion for the Cauchy problem (2).

Theorem 1. *Let* $u_j^0(x) \in W_q^1(R_3)$, $j = 1,2$, $u_3^0(x) \in W_q^1(R_3) \cap L_1(R_3)$, *and let the compatibility condition*

$$\operatorname{div} u^0(x) = 0 \tag{3}$$

be satisfied. If $q > 3/2$ *then the Cauchy problem (2) has a unique solution*

$$u_j(t,x) \in W_{q,\gamma}^{1,1}(R_4^+),\ j = 1,2,3, \quad p(t,x) \in W_{q,\gamma}^{0,2}(R_4^+), \quad \gamma > 0, \tag{4}$$

and the following estimate is valid:

$$\sum_{j=1}^{3} \|u_j(t,x),\, W_{q,\gamma}^{1,1}(R_4^+)\| + \|p(t,x),\, W_{q,\gamma}^{0,2}(R_4^+)\|$$

$$\le c\left(\sum_{j=1}^{3} \|u_j^0(x),\, W_q^1(R_3)\| + \|u_3^0(x),\, L_1(R_3)\|\right),$$

where the constant $c > 0$ *is independent of* $u^0(x)$.

The question is whether the restriction ($q > 3/2$) on the summand exponent is essential or not. We answer to the question in the present paper.

Theorem 2. *Let* $u_j^0(x) \in W_q^1(R_3)$, $j = 1,2$, $u_3^0(x) \in W_q^1(R_3) \cap L_{1,-1}(R_3)$, *and let condition (3) be satisfied. The Cauchy problem (2) has a unique solution (4) for* $1 < q \le 3/2$ *if and only if*

$$\int_{R_3} u_3^0(x)dx = 0. \tag{5}$$

Note that, for the first time, the necessity of additional requirements on data was discovered by S. A. Gal'pern [3] while constructing the L_2-theory of the Cauchy problem for a class of systems not of Cauchy – Kovalevsky type. Analogous peculiarities for mixed boundary value problems were observed by G. V. Demidenko [4].

In section 3, we construct a sequence of approximate solutions of the Cauchy problem (2). In section 4, we establish L_q-estimates for the sequence and prove Theorem 2. Deriving the estimates we follow the scheme proposed by G. V. Demidenko in [4] and described in detail in [1].

3. Approximate solutions

Consider the following algebraic system with the parameters $\tau = \gamma + i\eta \in C$, $\gamma > 0$, $\xi \in R_3 \backslash \{0\}$

$$
\begin{aligned}
&\tau v_1 + i\xi_1 v_4 = f_1(\xi),\\
&\tau v_2 + i\xi_2 v_4 = f_2(\xi),\\
&\tau v_3 + \omega^2 \tau^{-1} v_3 + i\xi_3 v_4 = f_3(\xi),\\
&i\xi_1 v_1 + i\xi_2 v_2 + i\xi_3 v_3 = 0.
\end{aligned} \tag{6}
$$

Suppose that the functions $f_j(\xi)$, $j = 1, 2, 3$, satisfy the condition

$$
i\xi_1 f_1(\xi) + i\xi_2 f_2(\xi) + i\xi_3 f_3(\xi) = 0. \tag{7}
$$

System (6) and condition (7) result from formal application of the Fourier transform in x and the Laplace transform in t to the Cauchy problem (2) and condition (3). It is obvious that a solution of (6) can be written as follows

$$
\begin{aligned}
&v_1(\tau, \xi) = -\frac{1}{\tau} i\xi_1 v_4(\tau, \xi) + \frac{1}{\tau} f_1(\xi),\\
&v_2(\tau, \xi) = -\frac{1}{\tau} i\xi_2 v_4(\tau, \xi) + \frac{1}{\tau} f_2(\xi),\\
&v_3(\tau, \xi) = -\frac{\tau}{\tau^2 + \omega^2} i\xi_3 v_4(\tau, \xi) + \frac{\tau}{\tau^2 + \omega^2} f_3(\xi),
\end{aligned} \tag{8}
$$

$$
v_4(\tau, \xi) = \frac{i\omega^2 \xi_3}{\tau^2 |\xi|^2 + \omega^2(\xi_1^2 + \xi_2^2)} f_3(\xi). \tag{9}
$$

Since $v_j(\tau, \xi)$, $j = 1, \dots, 4$, are analytic and bounded functions of τ, $\operatorname{Re} \tau = \gamma > 0$, by Theorem 5.2 of [1, Chapter 1] the functions

$$
w_j(t, \xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{(i\eta + \gamma)t} v_j(i\eta + \gamma, \xi)\, d\eta, \quad j = 1, \dots, 4,
$$

do not depend on $\gamma > 0$.

Now we construct an approximate solution of the Cauchy problem (2). We could obtain a formal solution of the problem by applying the inverse Fourier transform in x to the functions $w_j(t, \xi)$, $j = 1, \dots, 4$. However, the function $w_4(t, \xi)$ has nonintegrable singularity at $\xi = 0$. Therefore, it is necessary to regularize the inverse

Fourier operator. For this purpose we use the integral representation proposed by S. V. Uspenskiĭ [5] for functions $f(x) \in L_q(R_n)$:

$$(10)\qquad f(x) = \lim_{h\to 0}(2\pi)^{-n}\int\limits_h^{h^{-1}} v^{-1}\int\limits_{R_n}\int\limits_{R_n} e^{i(x-y)\xi}G(\xi v)f(y)\,d\xi\,dy\,dv,$$

where $G(\xi) = 2N|\xi|^{2N}\exp(-|\xi|^{2N})$, N is an arbitrary natural number (see obtaining the representation in [1, Chapter 1]).

Denote the Fourier transform of $u_3^0(x)$ by $\hat{u}_3^0(\xi)$. Let $f_3(\xi) = (2\pi)^{-1/2}\hat{u}_3^0(\xi)$. Define the functions

$$(11)\qquad p_h(t,x) = (2\pi)^{-3/2}\int\limits_h^{h^{-1}} v^{-1}\int\limits_{R_3} e^{i\xi x}G(\xi v)w_4(t,\xi)\,d\xi\,dv$$

$$= (2\pi)^{-2}\int\limits_h^{h^{-1}} v^{-1}\int\limits_{R_3}\int\limits_{R_1} e^{(i\eta+\gamma)t+i\xi x}G(\xi v) \times\frac{i\omega^2\xi_3}{\tau^2|\xi|^2+\omega^2(\xi_1^2+\xi_2^2)}\hat{u}_3^0(\xi)d\eta\,d\xi\,dv,$$

$$u_{1,h}(t,x) = u_1^0(x) - \int\limits_0^t D_{x_1}p_h(s,x)\,ds,$$

$$(12)\qquad u_{2,h}(t,x) = u_2^0(x) - \int\limits_0^t D_{x_2}p_h(s,x)\,ds,$$

$$u_{3,h}(t,x) = \cos(\omega t)u_3^0(x) - \int\limits_0^t \cos(\omega(t-s))D_{x_3}p_h(s,x)\,ds.$$

It is easy to prove that

$$\begin{aligned} &D_t u_{1,h} + D_{x_1}p_h \equiv 0,\\ &D_t u_{2,h} + D_{x_2}p_h \equiv 0,\\ &D_t u_{3,h} + \omega^2\int_0^t u_{3,h}\,ds + D_{x_3}p_h \equiv 0,\\ &u_{j,h}|_{t=0} = u_j^0(x),\quad j=1,2,3, \end{aligned}$$

and, by representation (10),

$$\|D_{x_1}u_{1,h} + D_{x_2}u_{2,h} + D_{x_3}u_{3,h},\ L_q(R_3)\| \to 0,\quad h\to 0.$$

Therefore, we can consider the vector-function $(u_h(t,x), p_h(t,x))$ as an approximate solution of the Cauchy problem (2).

4. Necessary and sufficient conditions of solvability

In this section we prove that condition (5) is necessary and sufficient for the Cauchy problem (2) to be solvable in the weighted Sobolev space $W^{r_1,r_2}_{q,\gamma}(R^+_4)$ for $1 < q \le 3/2$.

First, we establish estimates of the approximate solution

$$(u_h(t,x),\ p_h(t,x)).$$

Lemma 1. *Let $u^0_j(x) \in W^1_q(R_3)$, $j = 1,2,3$, and let compatibility condition (3) be satisfied. Then,*

$$(13)\qquad \sum_{j=1}^{3} \|u_{j,h}(t,x),\, W^{1,1}_{q,\gamma}(R^+_4)\| + \sum_{k=1}^{3} \|D_{x_k} p_h(t,x),\, W^{0,1}_{q,\gamma}(R^+_4)\| \le c \sum_{j=1}^{3} \|u^0_j(x),\, W^1_q(R_3)\|, \quad q > 1,$$

where the constant $c > 0$ is independent of $u^0(x)$. Furthermore,

$$(14)\qquad \sum_{j=1}^{3} \|u_{j,h_1}(t,x) - u_{j,h_2}(t,x),\, W^{1,1}_{q,\gamma}(R^+_4)\| + \sum_{k=1}^{3} \|D_{x_k} p_{h_1}(t,x) - D_{x_k} p_{h_2}(t,x),\, W^{0,1}_{q,\gamma}(R^+_4)\| \to 0, \quad h_1,\ h_2 \to 0.$$

Proof. Now we prove that

$$(15)\qquad \sum_{k=1}^{3} \|D_{x_k} p_h(t,x),\, L_{q,\gamma}(R^+_4)\| \le c\|u^0_3(x),\, L_q(R_3)\|.$$

From (11) we obtain

$$e^{-\gamma t} D_{x_k} p_h(t,x) = -(2\pi)^{-2} \int_h^{h^{-1}} v^{-1} \int_{R_3} e^{i\xi x} G(\xi v) \times \int_{R_1} e^{i\eta t} \frac{\omega^2 \xi_3 \xi_k}{\tau^2 |\xi|^2 + \omega^2(\xi_1^2 + \xi_2^2)} \hat{u}^0_3(\xi)\, d\eta\, d\xi\, dv.$$

Using properties of the Fourier transform, we have

$$(16)\qquad e^{-\gamma t} D_{x_k} p_h(t,x) = c \int_h^{h^{-1}} v^{-1} \int_{R_3} \int_{R_3} e^{i(x-y)s} G(sv)$$

$$\times \left\{ \int\limits_{R_3} \int\limits_{R_1} e^{i\xi y + i\eta t} \frac{\omega^2 \xi_3 \xi_k}{\tau^2 |\xi|^2 + \omega^2(\xi_1^2 + \xi_2^2)} \hat{u}_3^0(\xi)\, d\eta\, d\xi \right\}\, ds\, dy\, dv.$$

Denote the expression in the curly braces by $P(t,y)$. Establish the following estimate

$$\|P(t,y),\, L_q(R_4^+)\| \le c\|u_3^0(x),\, L_q(R_3)\|. \tag{17}$$

Since

$$\int\limits_0^\infty e^{-\tau\zeta} d\zeta = 1/\tau, \quad \tau = \gamma + i\eta, \quad \gamma > 0,$$

the function $P(t,y)$ can be written as follows

$$P(t,y) = \int\limits_{R_4} e^{i\xi y + i\eta t} \mu(\eta,\xi) \int\limits_{R_4} e^{-i\eta\zeta - i\xi x} e^{-\gamma\zeta} \theta(\zeta) u_3^0(x)\, d\zeta\, dx\, d\eta\, d\xi,$$

where

$$\mu(\eta,\xi) = \tau \frac{\omega^2 \xi_3 \xi_k}{\tau^2|\xi|^2 + \omega^2(\xi_1^2 + \xi_2^2)},$$

$\theta(\zeta)$ is the Heaviside function. It is not hard to verify that the function $\mu(\eta,\xi)$ satisfies the conditions of Lizorkin's theorem on multipliers [6]. Hence,

$$\|P(t,y),\, L_q(R_4)\| \le c\|e^{-\gamma\zeta}\theta(\zeta)u_3^0(x),\, L_q(R_4)\| \le c\|u_3^0(x),\, L_q(R_3)\|$$

and (17) is proved. By integral representation (10) and estimate (17), from (16) we obtain (15).

Similarly, it is easily shown that

$$\sum_{k=1}^{3}\sum_{j=1}^{3} \|D^2_{x_k x_j} p_h(t,x),\, L_{q,\gamma}(R_4^+)\| \le c \sum_{l=1}^{3} \|D_{x_l} u_3^0(x),\, L_q(R_3)\|. \tag{18}$$

Estimate the functions $u_{j,h}(t,x)$, $j = 1,2,3$. Consider the following function

$$v(t,x) = \int\limits_0^t g(t-s) w(s,x)\, ds,$$

where $|g(t)| \le c$ for $t \ge 0$, the function $w(t,x)$ belongs to $L_{q,\gamma}(R_4^+)$. Now we prove that $v(t,x)$ belongs to the space $L_{q,\gamma}(R_4^+)$ too. Rewrite the function $v(t,x)$ as follows

$$e^{-\gamma t} v(t,x) = \int\limits_0^t e^{-\gamma(t-s)} g(t-s) e^{-\gamma s} w(s,x)\, ds$$

$$= \int_{-\infty}^{\infty} e^{-\gamma(t-s)}\theta(t-s)g(t-s)e^{-\gamma s}\theta(s)w(s,x)\,ds = f_1(t) * f_2(t,x),$$

where

$$f_1(t) = e^{-\gamma t}\theta(t)g(t), \quad f_2(t,x) = e^{-\gamma t}\theta(t)w(t,x).$$

From this representation we have

$$\|v(t,x),\, L_{q,\gamma}(R_4^+)\| \le \|v(t,x),\, L_{q,\gamma}(R_4)\|$$
$$= \|\|v(t,x),\, L_{q,\gamma}(R_1)\|,\, L_q(R_3)\|$$
$$= \|\|f_1(t) * f_2(t,x),\, L_q(R_1)\|,\, L_q(R_3)\|.$$

Applying Young's inequality, we obtain

$$\|v(t,x),\, L_{q,\gamma}(R_4^+)\| \le \|\|f_1(t),\, L_1(R_1)\|\|f_2(t,x),\, L_q(R_1)\|,\, L_q(R_3)\|$$
$$= \|e^{-\gamma t}g(t),\, L_1(R_1^+)\|\|w(t,x),\, L_{q,\gamma}(R_4^+)\| \le \frac{c}{\gamma}\|w(t,x),\, L_{q,\gamma}(R_4^+)\|.$$

Thus, taking into account (15) and (18), from (12) we obtain

$$\|u_{k,h}(t,x),\, W^{1,1}_{q,\gamma}(R_4^+)\| \le \|u_k^0(x),\, W^{1,1}_{q,\gamma}(R_4^+)\|$$
$$+\|D_{x_k}p_h(t,x),\, L_{q,\gamma}(R_4^+)\| + \|\int_0^t D_{x_k}p_h(s,x)\,ds,\, L_{q,\gamma}(R_4^+)\|$$
$$+\sum_{j=1}^{3}\|\int_0^t D_{x_jx_k}p_h(s,x)\,ds,\, L_{q,\gamma}(R_4^+)\| \le c\sum_{j=1}^{3}\|u_j^0(x),\, W_q^1(R_3)\|,$$
$$k=1,2,$$
$$\|u_{3,h}(t,x),\, W^{1,1}_{q,\gamma}(R_4^+)\| \le \|\cos(\omega t)u_3^0(x),\, W^{1,1}_{q,\gamma}(R_4^+)\|$$
$$+\|D_{x_3}p_h(t,x),\, L_{q,\gamma}(R_4^+)\|$$
$$+\omega\|\int_0^t \sin(\omega(t-s))D_{x_3}p_h(s,x)\,ds,\, L_{q,\gamma}(R_4^+)\|$$
$$+\|\int_0^t \cos(\omega(t-s))D_{x_3}p_h(s,x)\,ds,\, L_{q,\gamma}(R_4^+)\|$$
$$+\sum_{j=1}^{3}\|\int_0^t \cos(\omega(t-s))D_{x_jx_3}p_h(s,x)\,ds,\, L_{q,\gamma}(R_4^+)\|$$

$$\leq c\sum_{j=1}^{3}\|u_j^0(x),\, W_q^1(R_3)\|.$$

Estimate (13) is proved. The proof of (14) is carried out by the same scheme. □

Lemma 2. *Let $u_3^0(x) \in L_q(R_3) \cap L_{1,-1}(R_3)$ and let condition (5) be satisfied. Then,*

$$\|p_h(t,x),\, L_{q,\gamma}(R_4^+)\| \tag{19}$$

$$\leq c\Big(\|u_3^0(x),\, L_q(R_3)\| + \|u_3^0(x),\, L_{1,-1}(R_3)\|\Big),$$

where the constant $c > 0$ is independent of $u_3^0(x)$. Furthermore,

$$\|p_{h_1}(t,x) - p_{h_2}(t,x),\, L_{q,\gamma}(R_4^+)\| \to 0, \quad h_1,\ h_2 \to 0. \tag{20}$$

Proof. By (11) we have

$$e^{-\gamma t}p_h(t,x) = (2\pi)^{-2}\int_h^1 v^{-1}\int_{R_4} e^{i\eta t + i\xi x}G(\xi v)$$

$$\times\frac{i\omega^2\xi_3}{\tau^2|\xi|^2 + \omega^2(\xi_1^2+\xi_2^2)}\hat{u}_3^0(\xi)\,d\eta\,d\xi\,dv + (2\pi)^{-2}\int_1^{h^{-1}} v^{-1}\int_{R_4} e^{i\eta t + i\xi x}G(\xi v)$$

$$\times\frac{i\omega^2\xi_3}{\tau^2|\xi|^2 + \omega^2(\xi_1^2+\xi_2^2)}\hat{u}_3^0(\xi)\,d\eta\,d\xi\,dv = P_1(t,x) + P_2(t,x).$$

Consider the first summand. Rewrite it as follows

$$P_1(t,x) = (2\pi)^{-2}\int_h^1 v^{-1}\int_{R_4} e^{i\eta t + i\xi x}G(\xi v)$$

$$\times\tau\frac{i\omega^2\xi_3}{\tau^2|\xi|^2 + \omega^2(\xi_1^2+\xi_2^2)}\int_{R_4} e^{-i\eta\zeta - i\xi y}e^{-\gamma\zeta}\theta(\zeta)u_3^0(y)\,d\zeta\,dy\,d\eta\,d\xi\,dv.$$

It is not hard to verify that the function

$$\tau|\xi|\frac{i\omega^2\xi_3}{\tau^2|\xi|^2 + \omega^2(\xi_1^2+\xi_2^2)}$$

satisfies the conditions of Lizorkin's theorem on multipliers. Consequently,

$$\|P_1(t,x), L_q(R_4^+)\| \le c\| \int\limits_h^1 v^{-1} \int\limits_{R_3} e^{i\xi x} G(\xi v)|\xi|^{-1}\hat{u}_3^0(\xi)\, d\xi\, dv,\ L_q(R_3)\|.$$

Taking into account a formula for the inverse Fourier transform of a product, by Minkowski's inequality we obtain

$$\|P_1(t,x), L_q(R_4^+)\|$$
$$\le c\int\limits_h^1 v^{-1}\| \int\limits_{R_3}\int\limits_{R_3} e^{i(x-y)\xi} G(\xi v)|\xi|^{-1} u_3^0(y)\, d\xi\, dy,\ L_q(R_3)\|\, dv.$$

According to Young's inequality, we find that

$$\|P_1(t,x), L_q(R_4^+)\|$$
$$\le c\int\limits_h^1 v^{-1}\| \int\limits_{R_3} e^{i\xi x} G(\xi v)|\xi|^{-1} d\xi,\ L_1(R_3)\|\, dv\, \|u_3^0(x),\ L_q(R_3)\|.$$

Using the change of variables

$$s_j = \xi_j v, \quad j = 1,2,3, \qquad y_l = x_l/v, \quad l = 1,2,3,$$

we have

$$\|P_1(t,x), L_q(R_4^+)\|$$
$$\le c\int\limits_h^1 dv \| \int\limits_{R_3} e^{isy} G(s)|s|^{-1} ds,\ L_1(R_3)\|\, \|u_3^0(x),\ L_q(R_3)\|.$$

Choosing sufficiently large N in the definition of $G(s)$, we obtain

$$\|P_1(t,x), L_q(R_4^+)\| \le c\|u_3^0(x),\ L_q(R_3)\|. \tag{21}$$

Estimate the function $P_2(t,x)$. From (5) it follows that

$$\hat{u}_3^0(x) = -(2\pi)^{-3/2} \int\limits_0^1 \int\limits_{R_3} e^{-i\mu\xi y}(i\xi y) u_3^0(y)\, dy\, d\mu.$$

Arguing as above, it is easy to show that

$$\|P_2(t,x), L_q(R_4^+)\| \le c\sum_{k=1}^{3} \int\limits_1^{h^{-1}} v^{-1}$$

$$\times \| \int_{R_3} e^{i\xi x} G(\xi v) \frac{\xi_k}{|\xi|} d\xi,\ L_q(R_3) \|\, dv\, \| x_k u_3^0(x),\ L_1(R_3) \|.$$

Using the change of variables

$$s_j = \xi_j v, \quad j = 1,2,3, \qquad y_l = x_l / v, \quad l = 1,2,3,$$

we have

$$\| P_2(t,x), L_q(R_4^+) \| \le c \sum_{k=1}^{3} \int_1^{h^{-1}} v^{-1-3+3/q} dv$$

$$\times \| \int_{R_3} e^{isy} G(s) \frac{s_k}{|s|} ds,\ L_q(R_3) \|\, \| u_3^0(x),\ L_{1,-1}(R_3) \|.$$

Since the integral $\int_1^{\infty} v^{-4+3/q} dv$ converges for $q > 1$, choosing sufficiently large N in the definition of $G(s)$, we obtain

$$\| P_2(t,x), L_q(R_4^+) \| \le c \| u_3^0(x),\ L_{1,-1}(R_3) \|. \tag{22}$$

Estimate (19) follows from (21) and (22). In analogous way, one can prove (20). □

Taking into account (14) and (20), by completeness of the Sobolev space $W_{q,\gamma}^{r_1,r_2}(R_4^+)$, there exist functions $u_j(t,x) \in W_{q,\gamma}^{1,1}(R_4^+)$, $j = 1,2,3$, $p(t,x) \in W_{q,\gamma}^{0,2}(R_4^+)$ such that

$$\| u_{j,h}(t,x) - u_j(t,x),\ W_{q,\gamma}^{1,1}(R_4^+) \| \to 0,$$

$$\| p_h(t,x) - p(t,x),\ W_{q,\gamma}^{0,2}(R_4^+) \| \to 0$$

as $h \to 0$. Moreover, according to (13) and (19), it holds that

$$\sum_{j=1}^{3} \| u_j(t,x),\ W_{q,\gamma}^{1,1}(R_4^+) \| + \| p(t,x),\ W_{q,\gamma}^{0,2}(R_4^+) \|$$

$$\le c \left(\sum_{j=1}^{3} \| u_j^0(x),\ W_q^1(R_3) \| + \| u_3^0(x),\ L_{1,-1}(R_3) \| \right).$$

It is easily shown that the vector-function $(u(t,x), p(t,x))$ is a solution of the Cauchy problem (2).

Thus, we proved the first part of the assertion of Theorem 2; i.e., we showed that condition (5) is sufficient for the Cauchy problem (2) to have solution (4). As follows from Lemma 1,

$$u(t,x) \in W_{q,\gamma}^{1,1}(R_4^+), \quad \nabla p(t,x) \in W_{q,\gamma}^{0,1}(R_4^+)$$

for $q > 1$. Therefore, to obtain the second part of assertion of Theorem 2 we have to prove that condition (5) is necessary for the function $p(t, x)$ to belong to $L_{q,\gamma}(R_4^+)$ for $q \le 3/2$.

Lemma 3. *Let $u_3^0(x) \in L_q(R_3) \cap L_{1,-1}(R_3)$. For the function $p(t,x)$ to belong to $L_{q,\gamma}(R_4^+)$ for $q \le 3/2$ it is necessary to have*

$$\int\limits_{R_3} u_3^0(x)dx = 0.$$

Proof. Suppose that

$$\int\limits_{R_3} u_3^0(x)dx \neq 0$$

and $p(t, x) \in L_{q,\gamma}(R_4^+)$ for $q \le 3/2$. Since

$$\|p(t,x),\, L_{q,\gamma}(R_4^+)\| \le c < \infty,$$

then, by the Hausdorff–Young inequality, we have

$$\|\tilde{p}(\tau,\xi),\, L_{q'}(R_4)\| \le c < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

where

$$\tilde{p}(\tau,\xi) = (2\pi)^{-2} \int\limits_0^\infty \int\limits_{R_3} e^{-\tau t - i\xi x} p(t,x)\, dt\, dx, \quad \tau = i\eta + \gamma.$$

Hence,

$$\sup_{\varepsilon > 0} \int\limits_{-\infty}^{\infty} \int\limits_{\varepsilon < |\xi| < 1} |\tilde{p}(\tau,\xi)|^{q'} d\xi\, d\eta \le c.$$

Obviously, the functions

$$\tilde{u}_j(\tau,\xi) = (2\pi)^{-2} \int\limits_0^\infty \int\limits_{R_3} e^{-\tau t - i\xi x} u_j(t,x)\, dt\, dx, \quad j = 1,2,3,$$

and the function $\tilde{p}(\tau,\xi)$ satisfy system (6) for

$$f_j(\xi) = (2\pi)^{-2} \int\limits_{R_3} e^{-i\xi x} u_j^0(x)\, dx, \quad j = 1,2,3,$$

and are given by (8), (9). By Hadamard's lemma, the function $\hat{u}_3^0(\xi)$ can be represented as follows

$$\hat{u}_3^0(\xi) = \hat{u}_3^0(0) + \sum_{j=1}^{3} \xi_j U_j(\xi),$$

where

$$U_j(\xi) = (2\pi)^{-3/2} \int_0^1 \int_{R_3} e^{-i\xi x\mu}(-ix_j) u_3^0(x)\, dx\, d\mu.$$

Consequently,

$$\tilde{p}(\tau,\xi) = (2\pi)^{-1/2} \frac{i\omega^2 \xi_3}{\tau^2|\xi|^2 + \omega^2(\xi_1^2+\xi_2^2)} \hat{u}_3^0(0)$$

$$+(2\pi)^{-1/2} \frac{i\omega^2 \xi_3}{\tau^2|\xi|^2 + \omega^2(\xi_1^2+\xi_2^2)} \sum_{j=1}^{3} \xi_j U_j(\xi) = J_1 + J_2.$$

It is obvious that

$$\sup_{\varepsilon>0} \int_{-\infty}^{\infty} \int_{\varepsilon<|\xi|<1} \left| J_2 \right|^{q'} d\xi\, d\eta \le c\|u_3^0(x),\, L_{1,-1}(R_3)\|^{q'} \le c.$$

Hence,

$$\sup_{\varepsilon>0} \int_{-\infty}^{\infty} \int_{\varepsilon<|\xi|<1} |J_1|^{q'} d\xi\, d\eta \le c.$$

By definition,

$$|J_1| = (2\pi)^{-1/2} \frac{|\xi_3|}{|\xi|^2} \left| \frac{\omega^2}{\tau^2 + \omega^2(\xi_1^2+\xi_2^2)/|\xi|^2} \right| |\hat{u}_3^0(0)|$$

$$\ge (2\pi)^{-1/2} \frac{|\xi_3|}{|\xi|^2} \frac{\omega^2}{\gamma^2+\eta^2+\omega^2} |\hat{u}_3^0(0)|.$$

Then,

$$\sup_{\varepsilon>0} \int_{\varepsilon<|\xi|<1} \left(\frac{|\xi_3|}{|\xi|^2} \right)^{q'} |\hat{u}_3^0(0)|^{q'} d\xi \le c.$$

Since $q \leq 3/2$ it follows that the function $(|\xi_3|/|\xi|^2)^{q'}$ is not integrable in a neighborhood of $\xi = 0$. Therefore, if condition (5) is not fulfilled, then $\hat{u}_3^0(0) \neq 0$ and

$$\lim_{\varepsilon \to 0} \int\limits_{\varepsilon < |\xi| < 1} \left(\frac{|\xi_3|}{|\xi|^2} \right)^{q'} |\hat{u}_3^0(0)|^{q'} d\xi = \infty.$$

This contradiction concludes the proof. □

Thus, we proved that condition (5) is necessary and sufficient for the Cauchy problem (2) to have solution (4). We omit the uniqueness proof. It is carried out by the scheme described in detail in [1].

Theorem 2 is proved.

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