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A modification of the matrix sign function method

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Summary. This paper is devoted to the dichotomy problem for the matrix spectrum with respect to the imaginary axis. We consider a method of constructing approximate projections onto invariant subspaces of matrices. The method is a modification of the matrix sign function method. We prove also a theorem on a perturbation for the matrix spectrum.

Key words: matrix sign function method, projections onto invariant subspaces, matrix spectrum dichotomy, perturbation of the matrix spectrum

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1. Introduction

The problem of constructing approximate projections onto invariant subspaces of matrices is a very important problem of linear algebra and numerical mathematics. At present, there are some algorithms for constructing approximate projections (see, for example, [1–5]).

This paper is devoted to the dichotomy problem for the matrix spectrum with respect to the imaginary axis. The matrix sign function method is the most popular method of constructing approximate projections for solving this problem. This method has been the subject of numerous studies (see, for example, [6–13]).

In 1996, the first author proposed a new method for constructing approximate projections onto invariant subspaces of linear operators [14]. This method is functional and based on the following theorem [14, 15].

Theorem 1. *Let $T : B \to B$ be a linear continuous operator in a Banach space B, and let T have the inverse operator T^{-1}. Suppose that there is a projection $P : B \to B$ such that*

$$PT = TP, \qquad \|TP\| < 1, \qquad \|T^{-1}(I-P)\| < 1.$$

Then the operator $I - T$ has the continuous inverse one $(I-T)^{-1}$,

$$\begin{aligned}(I-T)^{-1} &= (I-TP)^{-1}P - (I - T^{-1}(I-P))^{-1}T^{-1}(I-P) \\ &= P + TP(I-TP)^{-1} - T^{-1}(I-P)(I-T^{-1}(I-P))^{-1}\end{aligned}$$

and

$$\begin{aligned}\|(I-T)^{-1} - P\| \le \|TP\|(1-\|TP\|)^{-1} \\ + \|T^{-1}(I-P)\|(1-\|T^{-1}(I-P)\|)^{-1}.\end{aligned} \tag{1.1}$$

Remark 1. As follows from the above estimate, if

$$\|TP\| \approx 0, \qquad \|T^{-1}(I-P)\| \approx 0,$$

then

$$(I-T)^{-1} \approx P.$$

Some applications of Theorem 1 to constructing approximate projections are presented in [16]. By Theorem 1, one can obtain a modification of the matrix sign function method [15]. In the present paper, we consider this modification of the matrix sign function method. By our means, such modification is very simple for computations.

2. Modification of the matrix sign function method

In this section we illustrate an application of Theorem 1 to the problem of constructing approximate projections onto invariant subspaces of matrices.

Let A be $N \times N$ matrix and I be the identity $N \times N$ matrix. Suppose that the matrix A has no purely imaginary eigenvalues. Eigenvalues of the matrix A are unknown. By P_- we denote the projection onto the maximal invariant subspace of A corresponding to eigenvalues lying in the left half-plane

$$C_- = \{\lambda \in C : \operatorname{Re} \lambda < 0\}.$$

By P_+ we denote the projection onto the maximal invariant subspace of A corresponding to eigenvalues lying in the right half-plane

$$C_+ = \{\lambda \in C : \mathrm{Re}\,\lambda > 0\}.$$

We suppose

$$P_- A = AP_-, \quad P_- + P_+ = I.$$

Throughout the paper, $||A||$ denotes the spectral norm of A.

Consider the sequence $\{U_k\}$, where

$$U_k = (\tau A + I)^k (\tau A - I)^{-k}, \tag{2.1}$$

$$\begin{array}{lll} \tau = 1/2 & \text{if} & ||A|| \leq 1, \\ \tau = (2||A||)^{-1} & \text{if} & ||A|| > 1. \end{array}$$

Using M.G.Krein's lemma on "W-dissipative" operators [17], one can show that

$$||U_k P_-|| \to 0, \qquad ||U_k^{-1}(I - P_-)|| \to 0, \qquad k \to \infty.$$

Hence, by Theorem 1, for any sufficiently large $k \gg 1$ there exists an inverse matrix

$$(I - U_k)^{-1};$$

and

$$(I - U_k)^{-1} \to P_-, \quad (I - U_k^{-1})^{-1} \to P_+, \quad k \to \infty.$$

Then,

$$P_- \approx (I - U_k)^{-1}, \quad P_+ \approx (I - U_k^{-1})^{-1}, \quad k \gg 1. \tag{2.2}$$

Theorem 2. *Formulae (2.2) is a modification of the matrix sign function method.*

Proof. Using the matrix sign function method [1, 2], we have

$$P_- \approx \frac{1}{2}(I - X_l), \quad P_+ \approx \frac{1}{2}(I + X_l), \quad l \gg 1, \tag{2.3}$$

where

$$X_0 = \tau A, \quad X_l = \frac{1}{2}(X_{l-1} + X_{l-1}^{-1}), \quad l = 1, 2, ...$$

By the definition of the sequence $\{X_l\}$, we obtain

$$2(X_l + I)X_{l-1} = (X_{l-1} + I)^2, \tag{2.4}$$

$$2(X_l - I)X_{l-1} = (X_{l-1} - I)^2. \tag{2.5}$$

According to (2.4),

$$2(X_l+I)(X_l-I)X_{l-1}=(X_l-I)(X_{l-1}+I)^2.$$

Taking into account (2.5), we have

$$(X_l+I)(X_{l-1}-I)^2=(X_l-I)(X_{l-1}+I)^2. \tag{2.6}$$

Similarly,

$$2(X_{l-1}+I)X_{l-2}=(X_{l-2}+I)^2, \tag{2.7}$$

$$2(X_{l-1}-I)X_{l-2}=(X_{l-2}-I)^2, \quad l=2,3,\ldots \tag{2.8}$$

Multiply both sides of (2.6) by $(2X_{l-2})^2$. Then,

$$(X_l+I)[2(X_{l-1}-I)X_{l-2}]^2=(X_l-I)[2(X_{l-1}+I)X_{l-2}]^2.$$

Consequently, by (2.7) and (2.8), we obtain

$$(X_l+I)(X_{l-2}-I)^{2^2}=(X_l-I)(X_{l-2}+I)^{2^2}.$$

In the same way, we have

$$(X_l+I)(X_0-I)^{2^l}=(X_l-I)(X_0+I)^{2^l}.$$

Since $X_0=\tau A$, it follows that $\|X_0\|\le 1/2$. Hence,

$$(X_l+I)=(X_l-I)(X_0+I)^{2^l}(X_0-I)^{-2^l}$$

or

$$(X_l+I)=(X_l-I)U_{2^l}.$$

Then,

$$X_l(I-U_{2^l})=-I-U_{2^l}.$$

Since for sufficiently large $l\gg 1$ the operator $I-U_{2^l}$ has an inverse one, we have

$$X_l=(U_{2^l}-I)^{-1}(I+U_{2^l}).$$

Rewrite X_l as follows

$$\begin{aligned} X_l&=(U_{2^l}-I)^{-1}(U_{2^l}-I+2I)\\ &=I+2(U_{2^l}-I)^{-1}=I-2(I-U_{2^l})^{-1}\end{aligned}$$

or

$$\frac{1}{2}(I-X_l)=(I-U_{2^l})^{-1}.$$

Therefore, formulae (2.2) for $k=2^l$ coincide with (2.3). $\square$

Corollary 1. *If $k = 2^l$ and l is sufficiently large, then approximate construction (2.2) coincides with approximate construction (2.3):*

$$P_- \approx (I - U_{2^l})^{-1} = \tfrac{1}{2}(I - X_l),$$

$$P_+ \approx (I - U_{2^l}^{-1})^{-1} = \tfrac{1}{2}(I + X_l).$$

Remark 2. When a matrix A has eigenvalues on the imaginary axis, its matrix sign function is not defined. But using Theorem 1, one can solve the trichotomy problem (see [15]).

Remark 3. By the definition of U_k, it is necessary to calculate one inverse matrix only, and by the definition of X_l, it is necessary to calculate l inverse matrices.

3. Convergence rate of the sequence of approximate projections

We will estimate the convergence rate of the sequence of approximate projections

$$(I - U_k)^{-1} \to P_-, \quad (I - U_k^{-1})^{-1} \to P_+, \quad k \to \infty, \tag{3.1}$$

where the matrices U_k are defined by (2.1). To do it we consider the Lyapunov type integrals

$$H_0^- = \int_0^\infty (e^{sA}P_-)^* e^{sA} P_- \, ds,$$

$$H_0^+ = \int_0^\infty (e^{-sA}P_+)^* e^{-sA} P_+ \, ds.$$

Obviously, the matrices H_0^-, H_0^+ are Hermitian,

$$H_0^- \geq 0, \quad H_0^+ \geq 0, \quad H = H_0^- + H_0^+ > 0,$$

and the equalities

$$H_0^- A + A^* H_0^- = -P_-^* P_-, \quad H_0^+ A + A^* H_0^+ = P_+^* P_+ \tag{3.2}$$

are true.

Using (3.2), one can show the following equalities

$$\begin{aligned} &(\tau A^* - I)^{-1}(\tau A^* + I) H_0^- (\tau A + I)(\tau A - I)^{-1} - H_0^- \\ &\qquad = -2\tau(\tau A^* - I)^{-1} P_-^* P_- (\tau A - I)^{-1}, \end{aligned} \tag{3.3}$$

$$(\tau A^* + I)^{-1}(\tau A^* - I)H_0^+(\tau A - I)(\tau A + I)^{-1} - H_0^+ \tag{3.4}$$
$$= -2\tau(\tau A^* + I)^{-1}P_+^* P_+(\tau A + I)^{-1}.$$

Indeed, rewrite the left hand-side of (3.3) as follows

$$(\tau A^* - I)^{-1}(\tau A^* + I)H_0^-(\tau A + I)(\tau A - I)^{-1} - H_0^-$$
$$= (\tau A^* - I)^{-1}\left((\tau A^* + I)H_0^-(\tau A + I)\right.$$
$$\left. - \ (\tau A^* - I)H_0^-(\tau A - I)\right)(\tau A - I)^{-1}$$
$$= (\tau A^* - I)^{-1}\Big(\tau^2 A^* H_0^- A + \tau H_0^- A + \tau A^* H_0^- + H_0^-$$
$$- \ \tau^2 A^* H_0^- A + \tau H_0^- A + \tau A^* H_0^- - H_0^-\Big)(\tau A - I)^{-1}$$
$$= (\tau A^* - I)^{-1}\left(2\tau(H_0^- A + A^* H_0^-)\right)(\tau A - I)^{-1}.$$

Taking into account (3.2), we have (3.3). Similarly, we can obtain (3.4).

Determine the matrices

$$S_- = (\tau A^* - I)^{-1}(\tau A - I)^{-1}, \quad S_+ = (\tau A^* + I)^{-1}(\tau A + I)^{-1}.$$

The matrices are Hermitian positive definite. Then,

$$s_- = \min_{|u|=1}\langle S_- u, u\rangle > 0, \quad s_+ = \min_{|u|=1}\langle S_+ u, u\rangle > 0.$$

Hence, for any vector $u \in E_N$ we have

$$\langle S_- u, u\rangle \geq s|u|^2, \quad \langle S_+ u, u\rangle \geq s|u|^2, \tag{3.5}$$

where $s = \min\{s_-, s_+\}$.

Note that, using (2.1) and the property $AP_- = P_- A$, equalities (3.3) and (3.4) can be written as follows

$$U_1^* H_0^- U_1 - H_0^- = -2\tau P_-^* S_- P_-,$$
$$(U_1^{-1})^* H_0^+ U_1^{-1} - H_0^+ = -2\tau P_+^* S_+ P_+.$$

By $P_-^2 = P_-$, $P_+^2 = P_+$, for any vector $v \in E_N$ this yields

$$\langle H_0^- U_1 P_- v, U_1 P_- v\rangle = \langle H_0^- P_- v, P_- v\rangle - 2\tau\langle S_- P_- v, P_- v\rangle,$$
$$\langle H_0^+ U_1^{-1} P_+ v, U_1^{-1} P_+ v\rangle = \langle H_0^+ P_+ v, P_+ v\rangle - 2\tau\langle S_+ P_+ v, P_+ v\rangle.$$

Rewrite the equalities by using the Hermitian positive definite matrix

$$H = H_0^- + H_0^+.$$

Since

$$HP_- = H_0^- P_-, \quad HP_+ = H_0^+ P_+,$$

$$U_1 P_- = P_- U_1, \quad U_1^{-1} P_+ = P_+ U_1^{-1},$$

it follows that

$$\langle HU_1P_-v, U_1P_-v\rangle = \langle HP_-v, P_-v\rangle - 2\tau\langle S_-P_-v, P_-v\rangle, \tag{3.6}$$

$$\langle HU_1^{-1}P_+v, U_1^{-1}P_+v\rangle = \langle HP_+v, P_+v\rangle - 2\tau\langle S_+P_+v, P_+v\rangle. \tag{3.7}$$

Taking into account (3.5) and the estimate

$$\frac{1}{\|H\|}\langle Hu, u\rangle \le |u|^2, \quad u \in E_N, \tag{3.8}$$

we obtain

$$\langle HU_1P_-v, U_1P_-v\rangle \le \left(1 - \frac{2\tau s}{\|H\|}\right)\langle HP_-v, P_-v\rangle, \tag{3.9}$$

$$\langle HU_1^{-1}P_+v, U_1^{-1}P_+v\rangle \le \left(1 - \frac{2\tau s}{\|H\|}\right)\langle HP_+v, P_+v\rangle. \tag{3.10}$$

Since the matrices

$$H, \quad U_1^* H U_1, \quad (U_1^{-1})^* H U_1^{-1}$$

are Hermitian positive definite and $P_- + P_+ = I$, it follows that

$$q = 1 - \frac{2\tau s}{\|H\|} > 0. \tag{3.11}$$

The following theorem gives estimates of the convergence rate in (3.1).

Theorem 3. *There exists a natural number k_0 such that for $k \ge k_0$ the following estimates hold*

$$\|P_- - (I - U_k)^{-1}\| \le \alpha_k^-(1 - \alpha_k^-)^{-1} + \alpha_k^+(1 - \alpha_k^+)^{-1}, \tag{3.12}$$

$$\|P_+ - (I - U_k^{-1})^{-1}\| \le \alpha_k^-(1 - \alpha_k^-)^{-1} + \alpha_k^+(1 - \alpha_k^+)^{-1}, \tag{3.13}$$

where

$$\alpha_k^- = \sqrt{\operatorname{cond} H}\,\|P_-\|q^{k/2}, \quad \alpha_k^+ = \sqrt{\operatorname{cond} H}\,\|P_+\|q^{k/2},$$

$$\operatorname{cond} H = \|H\|\|H^{-1}\|,$$

and $q \in (0, 1)$ is defined by (3.11).

Proof. By (2.1) and (3.6), we have

$$\langle HU_kP_-v,\, U_kP_-v\rangle = \langle HU_{k-1}P_-v,\, U_{k-1}P_-v\rangle$$
$$-2\tau\langle S_-U_{k-1}P_-v,\, U_{k-1}P_-v\rangle, \quad v\in E_N, \quad k\geq 1.$$

Taking into account (3.5) and (3.8), we obtain

$$\langle HU_kP_-v,\, U_kP_-v\rangle \leq \left(1-\frac{2\tau s}{\|H\|}\right)\langle HU_{k-1}P_-v,\, U_{k-1}P_-v\rangle.$$

Hence, for any natural k the estimate

$$\langle HU_kP_-v,\, U_kP_-v\rangle \leq q^{k-1}\langle HU_1P_-v,\, U_1P_-v\rangle$$

holds. By (3.9),

$$\langle HU_kP_-v,\, U_kP_-v\rangle \leq q^k\|H\|\|P_-\|^2|v|^2.$$

Using (3.7) and (3.10), in the same way one can prove the estimate

$$\langle HU_k^{-1}P_+v,\, U_k^{-1}P_+v\rangle \leq q^k\|H\|\|P_+\|^2|v|^2.$$

From these inequalities we have

$$|U_kP_-v|^2 \leq q^k\|H^{-1}\|\|H\|\|P_-\|^2|v|^2,$$
$$|U_k^{-1}P_+v|^2 \leq q^k\|H^{-1}\|\|H\|\|P_+\|^2|v|^2, \quad v\in E_N.$$

Hence,

$$\|U_kP_-\| \leq q^{k/2}\sqrt{\operatorname{cond} H}\|P_-\| = \alpha_k^-,$$
$$\|U_k^{-1}P_+\| \leq q^{k/2}\sqrt{\operatorname{cond} H}\|P_+\| = \alpha_k^+.$$

By (3.11), there exists k_0 such that $\alpha_k^- < 1$, $\alpha_k^+ < 1$ for any $k\geq k_0$. Consequently, from (1.1) we obtain (3.12) and (3.13). □

4. Perturbation of the matrix spectrum

In this section we give conditions on matrix perturbations. The conditions are used for numerical solving of the dichotomy problem (see analogous results in [4, 7]).

Theorem 4. *Let the spectrum of the matrix A do not cross with the imaginary axis. If for a matrix A_1 the inequality*

$$(4.1)\qquad 2\|A_1\|\left(\|H_0^-\|\sqrt{2\|A\|\|H_0^-\|}+\|H_0^+\|\sqrt{2\|A\|\|H_0^+\|}\right) < 1$$

is true, then the spectrum of the matrix $A+A_1$ does not cross with the imaginary axis too. Moreover, the number of eigenvalues of $A+A_1$ in the left half-plane C_- equals the number of eigenvalues of A in the left half-plane.

Proof. Consider the system of ordinary differential equations on the real axis

$$\frac{du}{dt} = Bu + f(t), \quad t \in R. \tag{4.2}$$

According to properties of the Fourier transform, the system has a solution $u(t)$ in the Sobolev space $W_2^1(R)$ for any vector function $f(t) \in L_2(R)$ if and only if the linear system

$$(i\xi I \ - \ B)\widehat{u} \ = \ \widehat{f}(\xi), \quad \xi \in R,$$

has a solution $\widehat{u}(\xi)$, $\xi\widehat{u}(\xi) \in L_2(R)$ for any $\widehat{f}(\xi) \in L_2(R)$. The system has such solution $\widehat{u}(\xi)$ if and only if the matrix B has no purely imaginary eigenvalues. Obviously, the solution $u(t) \in W_2^1(R)$ is unique.

By the conditions of the theorem, if $B = A$, then for any $f(t) \in L_2(R)$ system (4.2) has a unique solution $u(t) \in W_2^1(R)$ and the following representation

$$u(t) = Rf(t) = \int_{-\infty}^{t} e^{(t-s)A} P_- f(s)ds - \int_{t}^{\infty} e^{(t-s)A} P_+ f(s)ds \tag{4.3}$$

is true, where P_- is the projection onto the maximal invariant subspace of the matrix A corresponding to the eigenvalues lying in the left half-plane, $P_- A = AP_-$, $P_- + P_+ = I$. Hence, it is enough to prove that the system

$$\frac{du}{dt} = (A + A_1)u + F(t), \quad t \in R, \tag{4.4}$$

has a solution $u(t) \in W_2^1(R)$ for an arbitrary vector function $F(t) \in L_2(R)$.

We will construct a solution of system (4.4) in the form

$$u(t) = Rf(t), \quad f(t) \in L_2(R),$$

where the operator R is defined by (4.3). Obviously, a vector function $f(t)$ must be a solution of the integral equation

$$f(t) - A_1 Rf(t) = F(t). \tag{4.5}$$

Show the estimate

$$\|A_1 Rf(t), L_2(R)\| \le q\|f(t), L_2(R)\|, \quad q < 1. \tag{4.6}$$

Using the Heaviside function $\theta(t)$, the expression $A_1Rf(t)$ can be written as follows

$$A_1Rf(t) = A_1\int_{-\infty}^{\infty}\theta(t-s)e^{(t-s)A}P_-f(s)ds$$

$$-A_1\int_{-\infty}^{\infty}\theta(s-t)e^{(t-s)A}P_+f(s)ds.$$

By Minkowski's and Young's inequalities, we have

$$\|A_1Rf(t),\, L_2(R)\| \le \|A_1\|\|\int_{-\infty}^{\infty}\|\theta(t-s)e^{(t-s)A}P_-\||f(s)|ds,\, L_2(R)\|$$

$$+\|A_1\|\|\int_{-\infty}^{\infty}\|\theta(s-t)e^{(t-s)A}P_+\||f(s)|ds,\, L_2(R)\|$$

$$\le \|A_1\|\int_0^{\infty}\|e^{tA}P_-\|dt\|f(s),\, L_2(R)\|$$

$$+\|A_1\|\int_{-\infty}^{0}\|e^{tA}P_+\|dt\|f(s),\, L_2(R)\|.$$

Taking into account the estimates [4, 5]:

$$\|e^{tA}P_-\| \le \sqrt{2\|A\|\|H_0^-\|}e^{-t/(2\|H_0^-\|)},$$

$$\|e^{-tA}P_+\| \le \sqrt{2\|A\|\|H_0^+\|}e^{-t/(2\|H_0^+\|)}, \quad t>0,$$

we obtain

$$\|A_1Rf(t),\, L_2(R)\| \le 2\|A_1\|\Big(\|H_0^-\|\sqrt{2\|A\|\|H_0^-\|}$$

$$+\|H_0^+\|\sqrt{2\|A\|\|H_0^+\|}\Big)\|f(s),\, L_2(R)\|.$$

By condition (4.1), we have estimate (4.6).

According to (4.6), equation (4.5) has a unique solution in the space $L_2(R)$

$$f(t) = (I - A_1R)^{-1}F(t)$$

for any $F(t) \in L_2(R)$. Hence, we constructed the solution of (4.4)

$$u(t) = R(I - A_1R)^{-1}F(t) \in W_2^1(R).$$

Consequently, the matrix $A + A_1$ has no purely imaginary eigenvalues and the number of its eigenvalues in the left half-plane C_- equals the number of eigenvalues of the matrix A in the left half-plane. □

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