Selçuk J. Appl. Math.
Vol. 14. No. 1. pp. 21-30, 2013

Selçuk Journal of Applied Mathematics

Approximating Common Fixed Point of a Finite Family with Errors for Generalized Asymptotically Quasi-Nonexpansive Mappings in Banach Spaces

Esra Yolacan[1], Hukmi Kızıltunc[2]

Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkiye.
e-mail: [2]hukmu@atauni.edu.tr, [1]yolacanesra@gmail.com

Received Date: March 10, 2011
Accepted Date: May 17, 2012

Abstract. In this paper, we define and study sufficient and necessary conditions for modified finite-step iterative sequences with mean errors for a finite family of generalized asymptotically quasi-nonexpansive mappings in real Banach spaces to converge to a common fixed point. The results of this paper can be viewed as an improvement and extension the corresponding results of [12], [13] and others.

Key words: Generalized asymptotically quasi-nonexpansive mapping; Common fixed point; Modified iterative process; Strong convergence; Banach space.
AMS Classification: 47H10; 47H09.

1. Introduction and Preliminaries

Let K be a nonempty subset of Banach space E. A mapping $T : K \to K$ is said to be

(a) *asymptotically nonexpansive* [1] if there exists a sequence $\{r_n\}$ in $[0, \infty)$ such that $r_n \to 0$ and

$$\|T^n x - T^n y\| \le (1 + r_n) \|x - y\| \tag{1}$$

for all $x, y \in K$ and $n \ge 1$;

(b) *asymptotically quasi − nonexpansive* [2] if $F(T) := \{p \in K : Tp = p\} \neq \varnothing$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ such that $r_n \to 0$ and

$$\|T^n x - p\| \le (1 + r_n) \|x - p\| \tag{2}$$

[2]Corresponding Author

for all $x \in K$, $p \in F(T)$ and $n \geq 1$;

(c) *generalized asymptotically nonexpansive* [4]if there exist sequences $\{r_n\}$, $\{l_n\}$ in $[0, \infty)$ such that $r_n, l_n \to 0$ and

$$\|T^n x - T^n y\| \leq (1 + r_n) \|x - y\| + l_n \tag{3}$$

for all $x, y \in K$ and $n \geq 1$;

(d) *generalized asymptotically quasi−nonexpansive* [3] if $F(T) \neq \varnothing$ and there exist sequences $\{r_n\}$, $\{l_n\}$ in $[0, \infty)$ such that $r_n, l_n \to 0$

$$\|T^n x - p\| \leq (1 + r_n) \|x - p\| + l_n \tag{4}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

It is well known that the concept of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] who proved that every asymptotically nonexpansive self-mapping of nonempty closed bounded and convex subset of a uniformly convex Banach space has fixed point. In 2003, Zhou et al. [4] introduced a new class of generalized asymptotically nonexpansive mapping and gave a necessary and sufficient condition for the modified Ishikawa and Mann iterative sequences to converge to fixed points for the class of mappings. Atsushiba [5] studied the necessary and sufficient condition for the convergence of iterative sequences to a common fixed point of the finite family of asymptotically nonexpansive mappings in Banach spaces. Suzuki [6], Zeng and Yao [7] discussed a necessary and sufficient condition for common fixed points of two nonexpansive mappings and a finite family of nonexpansive mappings, and proved some convergence theorems for approximating a common fixed point, respectively.

In 2006, Lan [8] introduced a new class of generalized asymptotically quasi-nonexpansive mappings and gave necessary and sufficient condition for the 2-step modified Ishikawa iterative sequences to converge to fixed points for the class of mappings. In 2007, Yang [17] established convergence theorems for the modified multistep iterative process for some common fixed point of a finite family of nonself asymptotically nonexpansive mappings. In 2008, Nantadilok [9] extension and improvement the result of Lan [8] and gave a necessary and sufficient condition for convergence of common fixed point for three-step iterative sequence with errors for generalized asymptotically quasi-nonexpansive mappings. Lan [8] and many authors (e.g., [9], [10], [11]) have investigated convergence theorems for such mappings without awareness that Lan's mappings are not new ones. In 2009, Suantai et al. [12] and Saejung [13] introduced a general iteration scheme for a finite family of generalized asymptotically quasi-nonexpansive mappings in Banach spaces.

Inspired and motivated by this facts, we defined and study the convergence theorems of finite steps iterative sequences with errors for generalized asymptotically quasi-nonexpansive mappings. The scheme (5) is defined as follows:

Let E be a normed space, K be a nonempty closed convex subset of E. Let $T_i : K \to K$ $(i = 1, 2, \ldots, k)$ be mappings and $\mathcal{F} := \cap_{i=1}^{k} F(T_i) \neq \varnothing$. Then for a given $x_1 \in K$ and $n \geq 1$, compute the iterative sequences $\{x_n\}, \{y_n\}, \ldots, \{y_{n+k-2}\}$ defined by
(5)

$$
\begin{aligned}
y_n &= \alpha_{nk} x_n + \delta_{nk} T_k^n x_n + \gamma_{nk} u_{nk}, \\
y_{n+1} &= \alpha_{n(k-1)} x_n + \delta_{n(k-1)} T_{k-1}^n y_n + \beta_{n(k-1)} T_{k-1}^n x_n + \gamma_{n(k-1)} u_{n(k-1)}, \\
y_{n+2} &= \alpha_{n(k-2)} x_n + \delta_{n(k-2)} T_{k-2}^n y_{n+1} + \beta_{n(k-2)} T_{k-2}^n y_n + \gamma_{n(k-2)} u_{n(k-2)}, \\
&\vdots \\
y_{n+k-2} &= \alpha_{n2} x_n + \delta_{n2} T_2^n y_{n+k-3} + \beta_{n2} T_2^n y_{n+k-4} + \gamma_{n2} u_{n2}, \\
x_{n+1} &= \alpha_{n1} x_n + \delta_{n1} T_1^n y_{n+k-2} + \beta_{n1} T_1^n y_{n+k-3} + \gamma_{n1} u_{n1},
\end{aligned}
$$

where $\{u_{n1}\}, \{u_{n2}\}, \ldots, \{u_{nk}\}$ are bounded sequences in K with $\{\alpha_{ni}\}, \{\delta_{ni}\}$, $\{\beta_{ni}\}$ and $\{\gamma_{ni}\}$ are appropriate real sequences in $[0, 1]$ such that $\alpha_{nk} + \delta_{nk} + \gamma_{nk} = 1$ and $\alpha_{ni} + \delta_{ni} + \beta_{ni} + \gamma_{ni} = 1$ for all $i = 1, 2, \ldots, k-1$ and all n.
The purpose of this paper is to study the convergence theorems of finite steps iterative sequences with errors for generalized asymptotically quasi-nonexpansive mappings in Banach spaces.
In the sequel, the following lemmas are needed to prove our main results.
A family $\{T_i : i = 1, 2, \ldots, k\}$ of self-mappings of K with $\mathcal{F} := \cap_{i=1}^{k} F(T_i) \neq \varnothing$ is said to satisfy the following conditions.

(1) *Condition* $(\overline{A})$ [14]. If there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\frac{1}{k} \sum_{i=1}^{k} \|x - T_i x\| \geq f(d(x, \mathcal{F}))$ for all $x \in K$, where $d(x, \mathcal{F}) = \inf\{\|x - p\| : p \in \mathcal{F}\}$.

(2) *Condition* $(\overline{B})$ [14]. If there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\max_{1 \leq i \leq k} \{\|x - T_i x\|\} \geq f(d(x, \mathcal{F}))$ for all $x \in K$.

(3) *Condition* $(\overline{C})$ [14]. If there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - T_i x\| \geq f(d(x, \mathcal{F}))$ for all $x \in K$ and for at least one T_i, $i = 1, 2, \ldots, k$.

Note that $(\overline{B})$ and $(\overline{C})$ are equivalent, condition $(\overline{B})$ reduces to condition (I) when all but one of T_i's are identities, and in addition, it also condition $(\overline{A})$.
It is well known that every continuous and demicompact mapping must satisfy condition (I) (see [15]). Since every completely continuous $T : K \to K$ is continuous and demicompact so that it satisfies condition (I). Thus we will use condition $(\overline{C})$ instead of the demicompactness and complete continuity of a family $\{T_i : i = 1, 2, \ldots, k\}$.

Lemma 1. [16] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \le (1 + c_n)\, a_n + b_n, \quad n \ge 1, \tag{6}$$

if $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$,

(i) then $\lim\limits_{n\to\infty} a_n$ exists;

(ii) $\lim\limits_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

2. Main Results

Our first result is the strong convergence theorems of the iterative scheme (5) for a finite family of generalized asymptotically quasi-nonexpansive mappings in Banach space. In order that prove our main results, the following lemma is needed.

Lemma 2. Let E be a Banach space and K a nonempty closed and convex subset of E, and $\{T_i : i = 1, 2, \dots, k\}$ a finite family of generalized asymptotically quasi-nonexpansive self-mappings of K with the sequences $\{r_{ni}\}, \{l_{ni}\} \subset [0, \infty)$ such that $\sum\limits_{n=1}^{\infty} r_{ni} < \infty$ and $\sum\limits_{n=1}^{\infty} l_{ni} < \infty$ for all $i = 1, 2, \dots, k$. Assume that $\mathcal{F} \ne \varnothing$ and $\sum\limits_{n=1}^{\infty} \gamma_{ni} < \infty$ for each $i = 1, 2, \dots, k$. For a given $x_1 \in K$, let the sequences $\{x_n\}, \{y_n\}, \dots, \{y_{n+k-2}\}$ be defined by (5). Then

(a) there exist sequences $\{b_n\}$ and $\{c_{ni}\}$ in $[0, \infty)$ such that $\sum\limits_{n=1}^{\infty} b_n < \infty$, $\sum\limits_{n=1}^{\infty} c_{ni} < \infty$, and $\|y_{n+j} - p\| \le (1 + b_n)^{j+1} \|x_n - p\| + c_{n(j+1)}$ for all $j = 0, 1, \dots, k-2$ and all $p \in \mathcal{F}$;

(b) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$;

(c) there exist constant $M > 0$ and $\{s_i\}$ in $[0, \infty)$ such that $\sum\limits_{i=1}^{\infty} s_i < \infty$ and $\|x_{n+m} - p\| \le M \|x_n - p\| + \sum\limits_{i=n}^{\infty} s_i$ for all $p \in \mathcal{F}$ and n, $m \in \mathbb{N}$.

Proof. (a) Let $p \in \mathcal{F}$, $b_n = \max_{1\le i\le k} \{r_{ni}\}$ and $d_n = \max_{1\le i\le k} \{l_{ni}\}$ for all n.

Since $\sum\limits_{n=1}^{\infty} r_{ni} < \infty$ and $\sum\limits_{n=1}^{\infty} l_{ni} < \infty$ for all $i = 1, 2, \dots, k$, therefore $\sum\limits_{n=1}^{\infty} b_n < \infty$ and $\sum\limits_{n=1}^{\infty} d_n < \infty$. For each $n \ge 1$, we note that

$$\begin{aligned}
\|y_n - p\| &\le \|\alpha_{nk} x_n + \delta_{nk} T_k^n x_n + \gamma_{nk} u_{nk} - p\| \\
&\le \alpha_{nk} \|x_n - p\| + \delta_{nk} \|T_k^n x_n - p\| + \gamma_{nk} \|u_{nk} - p\| \\
&\le \alpha_{nk} \|x_n - p\| + \delta_{nk} (1 + r_{nk}) \|x_n - p\| + \delta_{nk} l_{nk} + \gamma_{nk} \|u_{nk} - p\| \\
&\le \alpha_{nk} (1 + b_n) \|x_n - p\| + \delta_{nk} (1 + b_n) \|x_n - p\| + \delta_{nk} d_n + \gamma_{nk} \|u_{nk} - p\| \\
&\le \alpha_{nk} (1 + b_n) \|x_n - p\| + \delta_{nk} (1 + b_n) \|x_n - p\| + \delta_{nk} d_n + \gamma_{nk} \|u_{nk} - p\|
\end{aligned} \tag{7}$$

where $c_{n1} = \delta_{nk} d_n + \gamma_{nk} \|u_{nk} - p\|$. Since $\{u_{nk}\}$ is bounded, $\sum_{n=1}^{\infty} \gamma_{nk} < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, we obtain that $\sum_{n=1}^{\infty} c_{n1} < \infty$. It follows from (7) that

(8)

$$
\begin{aligned}
\|y_{n+1} - p\| \leq & \Big\| \alpha_{n(k-1)} x_n + \delta_{n(k-1)} T_{k-1}^n y_n + \beta_{n(k-1)} T_{k-1}^n x_n \\
& + \gamma_{n(k-1)} u_{n(k-1)} - p \Big\| \\
\leq & \alpha_{n(k-1)} \|x_n - p\| + \delta_{n(k-1)} \left\| T_{k-1}^n y_n - p \right\| + \beta_{n(k-1)} \left\| T_{k-1}^n x_n - p \right\| \\
& + \gamma_{n(k-1)} \left\| u_{n(k-1)} - p \right\| \\
\leq & \alpha_{n(k-1)} \|x_n - p\| + \delta_{n(k-1)} (1 + b_n) \|y_n - p\| + \delta_{n(k-1)} d_n \\
& + \beta_{n(k-1)} (1 + b_n) \|x_n - p\| + \beta_{n(k-1)} d_n + \gamma_{n(k-1)} \left\| u_{n(k-1)} - p \right\| \\
\leq & \alpha_{n(k-1)} \|x_n - p\| + \delta_{n(k-1)} (1 + b_n) \left[(1 + b_n) \|x_n - p\| + c_{n1} \right] + \delta_{n(k-1)} d_n \\
& + \beta_{n(k-1)} (1 + b_n)^2 \|x_n - p\| + \beta_{n(k-1)} d_n + \gamma_{n(k-1)} \left\| u_{n(k-1)} - p \right\| \\
\leq & \left(\alpha_{n(k-1)} + \delta_{n(k-1)} + \beta_{n(k-1)} \right) (1 + b_n)^2 \|x_n - p\| + \delta_{n(k-1)} (1 + b_n) c_{n1} \\
& + \delta_{n(k-1)} d_n + \beta_{n(k-1)} d_n + \gamma_{n(k-1)} \left\| u_{n(k-1)} - p \right\| \\
\leq & (1 + b_n)^2 \|x_n - p\| + c_{n2},
\end{aligned}
$$

where $c_{n2} = \delta_{n(k-1)} (1 + b_n) c_{n1} + \delta_{n(k-1)} d_n + \beta_{n(k-1)} d_n + \gamma_{n(k-1)} \left\| u_{n(k-1)} - p \right\|$. Since $\left\{ u_{n(k-1)} \right\}$, $\{b_n\}$ are bounded, $\sum_{n=1}^{\infty} c_{n1} < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, and $\sum_{n=1}^{\infty} \gamma_{n(k-1)} < \infty$, it follows that $\sum_{n=1}^{\infty} c_{n2} < \infty$. Moreover, we see that

(9)

$$
\begin{aligned}
\|y_{n+2} - p\| \leq & \Big\| \alpha_{n(k-2)} x_n + \delta_{n(k-2)} T_{k-2}^n y_{n+1} + \beta_{n(k-2)} T_{k-2}^n y_n \\
& + \gamma_{n(k-2)} u_{n(k-2)} - p \Big\| \\
\leq & \alpha_{n(k-2)} \|x_n - p\| + \delta_{n(k-2)} \left\| T_{k-2}^n y_{n+1} - p \right\| + \beta_{n(k-2)} \left\| T_{k-2}^n y_n - p \right\| \\
& + \gamma_{n(k-2)} \left\| u_{n(k-2)} - p \right\| \\
\leq & \alpha_{n(k-2)} \|x_n - p\| + \delta_{n(k-2)} (1 + b_n) \|y_{n+1} - p\| + \delta_{n(k-2)} d_n \\
& + \beta_{n(k-2)} (1 + b_n) \|y_n - p\| + \beta_{n(k-2)} d_n + \gamma_{n(k-2)} \left\| u_{n(k-2)} - p \right\| \\
\leq & \alpha_{n(k-2)} \|x_n - p\| + \delta_{n(k-2)} (1 + b_n) \left[(1 + b_n)^2 \|x_n - p\| + c_{n2} \right] + \delta_{n(k-2)} d_n \\
& + \beta_{n(k-2)} (1 + b_n) \left[(1 + b_n) \|x_n - p\| + c_{n1} \right] + \beta_{n(k-2)} d_n + \gamma_{n(k-2)} \left\| u_{n(k-2)} - p \right\| \\
\leq & \left(\alpha_{n(k-2)} + \delta_{n(k-2)} + \beta_{n(k-2)} \right) (1 + b_n)^3 \|x_n - p\| + \delta_{n(k-2)} (1 + b_n) c_{n2} \\
& + \delta_{n(k-2)} d_n + \beta_{n(k-2)} (1 + b_n) c_{n1} + \beta_{n(k-2)} d_n + \gamma_{n(k-2)} \left\| u_{n(k-2)} - p \right\| \\
\leq & (1 + b_n)^3 \|x_n - p\| + c_{n3},
\end{aligned}
$$

where $c_{n3} = \delta_{n(k-2)} (1 + b_n) c_{n2} + \delta_{n(k-2)} d_n + \beta_{n(k-2)} (1 + b_n) c_{n1} + \beta_{n(k-2)} d_n +$

$\gamma_{n(k-2)} \left\| u_{n(k-2)} - p \right\|$. Since $\left\{ u_{n(k-2)} \right\}$, $\{b_n\}$ are bounded, $\sum\limits_{n=1}^{\infty} c_{n2} < \infty$, $\sum\limits_{n=1}^{\infty} d_n < \infty$, and $\sum\limits_{n=1}^{\infty} \gamma_{n(k-2)} < \infty$, it follows that $\sum\limits_{n=1}^{\infty} c_{n3} < \infty$. By continuing the above method, there are nonnegative real sequences $\{c_{ni}\}$ in $[0, \infty)$ such that $\sum\limits_{n=1}^{\infty} c_{ni} < \infty$ and

$$\|y_{n+j} - p\| \leq (1 + b_n)^{j+1} \|x_n - p\| + c_{n(j+1)}, \quad j = 0, 1, \ldots, k-2. \tag{10}$$

This completes the proof of (a).

(b) It follows from (5), (10) that

(11)

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \|\alpha_{n1} x_n + \delta_{n1} T_1^n y_{n+k-2} + \beta_{n1} T_1^n y_{n+k-3} + \gamma_{n1} u_{n1} - p\| \\
&\leq \alpha_{n1} \|x_n - p\| + \delta_{n1} \|T_1^n y_{n+k-2} - p\| + \beta_{n1} \|T_1^n y_{n+k-3} - p\| + \gamma_{n1} \|u_{n1} - p\| \\
&\leq \alpha_{n1} \|x_n - p\| + \delta_{n1} \left[(1 + b_n) \|y_{n+k-2} - p\| + d_n \right] \\
&\quad + \beta_{n1} \left[(1 + b_n) \|y_{n+k-3} - p\| + d_n \right] + \gamma_{n1} \|u_{n1} - p\| \\
&\leq \alpha_{n1} \|x_n - p\| + \delta_{n1} (1 + b_n) \|y_{n+k-2} - p\| + \delta_{n1} d_n \\
&\quad + \beta_{n1} (1 + b_n) \|y_{n+k-3} - p\| + \beta_{n1} d_n + \gamma_{n1} \|u_{n1} - p\| \\
&\leq \alpha_{n1} (1 + b_n)^k \|x_n - p\| + \delta_{n1} (1 + b_n) \left[(1 + b_n)^{k-1} \|x_n - p\| + c_{n(k-1)} \right] + \delta_{n1} d_n \\
&\quad + \beta_{n1} (1 + b_n) \left[(1 + b_n)^{k-2} \|x_n - p\| + c_{n(k-2)} \right] + \beta_{n1} d_n + \gamma_{n1} \|u_{n1} - p\| \\
&\leq (\alpha_{n1} + \delta_{n1} + \beta_{n1}) (1 + b_n)^k \|x_n - p\| + \delta_{n1} (1 + b_n) c_{n(k-1)} + \delta_{n1} d_n \\
&\quad + \beta_{n1} (1 + b_n) c_{n(k-2)} + \beta_{n1} d_n + \gamma_{n1} \|u_{n1} - p\| \\
&\leq (1 + b_n)^k \|x_n - p\| + c_{nk}
\end{aligned}$$

where $c_{nk} = \delta_{n1} (1 + b_n) c_{n(k-1)} + \delta_{n1} d_n + \beta_{n1} (1 + b_n) c_{n(k-2)} + \beta_{n1} d_n + \gamma_{n1} \|u_{n1} - p\|$. Since $\{u_{n1}\}$, $\{b_n\}$ are bounded, $\sum\limits_{n=1}^{\infty} c_{n(k-1)} < \infty$, $\sum\limits_{n=1}^{\infty} c_{n(k-2)} < \infty$, $\sum\limits_{n=1}^{\infty} d_n < \infty$, and $\sum\limits_{n=1}^{\infty} \gamma_{n1} < \infty$, it follows that $\sum\limits_{n=1}^{\infty} c_{nk} < \infty$. We have

$$\|x_{n+1} - p\| \leq (1 + b_n)^k \|x_n - p\| + c_{nk}. \tag{12}$$

It follows from Lemma 1 (i) that $\lim\limits_{n \to \infty} \|x_n - p\|$ exists, for all $p \in \mathcal{F}$.

(c) If $t \geq 0$, then $1 + t \leq e^t$ and so, $(1 + t)^k \leq e^{kt}$, for $k = 1, 2, \ldots$. Thus, from

(12), it follows that

$$
\begin{aligned}
\|x_{n+m}-p\| &\leq (1+b_{m+n-1})^k \|x_{n+m-1}-p\| + c_{(m+n-1)k} \\
&\leq \exp\{kb_{m+n-1}\} \|x_{n+m-1}-p\| + c_{(m+n-1)k} \\
&\;\;\vdots \\
&\leq \exp\left\{k \sum_{i=n}^{n+m-1} b_i\right\} \|x_n - p\| + \sum_{i=n}^{n+m-1} c_{in} \qquad (13)\\
&\leq \exp\left\{k \sum_{i=1}^{\infty} b_i\right\} \|x_n - p\| + \sum_{i=n}^{\infty} c_{in} \\
&\leq M \|x_n - p\| + \sum_{i=n}^{\infty} s_i,
\end{aligned}
$$

where $M = \exp\left\{k \sum_{i=n}^{\infty} b_i\right\}$ and $s_i = c_{in}$.

Theorem 1. Let E be a Banach space and K a nonempty closed and convex subset of E and $\{T_i : i = 1, 2, \ldots, k\}$ a finite family of generalized asymptotically quasi-nonexpansive self-mappings of K with the sequences $\{r_{ni}\}, \{l_{ni}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_{ni} < \infty$ and $\sum_{n=1}^{\infty} l_{ni} < \infty$ for all $i = 1, 2, \ldots, k$. Assume that $\mathcal{F} \neq \varnothing$ is closed and $\sum_{n=1}^{\infty} \gamma_{ni} < \infty$ for each $i = 1, 2, \ldots, k$. Then the iterative sequence $\{x_n\}, \{y_n\}, \ldots, \{y_{n+k-2}\}$ defined by (5) converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Proof. We prove only the sufficiency because the necessity is obvious. From (12), we have $\|x_{n+1} - p\| \leq (1 + b_n)^k \|x_n - p\| + c_{nk}$, for all n and all $p \in \mathcal{F}$. Hence, we have

$$
\begin{aligned}
d(x_{n+1}, \mathcal{F}) &\leq (1 + b_n)^k d(x_n, \mathcal{F}) + c_{nk} \\
&= \left(1 + \sum_{r=1}^{k} \tfrac{k(k-1)\ldots(k-r+1)}{r!} b_n^r\right) d(x_n, \mathcal{F}) + c_{nk}.
\end{aligned} \tag{14}
$$

Since $\sum_{n=1}^{\infty} b_n < \infty$, it follows that $\sum_{n=1}^{\infty} \sum_{r=1}^{k} \left(\tfrac{k(k-1)\ldots(k-r+1)}{r!}\right) b_n^r < \infty$. Since $\sum_{n=1}^{\infty} c_{nk} < \infty$ and $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$, it follows from Lemma 1 (ii) that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 2 (c), we have

$$
\|x_{n+m} - p\| \leq M \|x_n - p\| + \sum_{i=n}^{\infty} s_i, \qquad \forall p \in \mathcal{F}, \quad n, m \in \mathbb{N}. \tag{15}
$$

Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$ and $\sum_{i=1}^{\infty} s_i < \infty$, therefore for $\varepsilon > 0$, there exists

$\forall n_0 \in \mathbb{N}$ such that

$$d(x_n, \mathcal{F}) < \frac{\varepsilon}{4M}, \quad \sum_{i=n_0}^{\infty} s_i < \frac{\varepsilon}{4}, \quad \forall n \geq n_0. \tag{16}$$

Therefore, there exists q in $\mathcal{F}$ such that

$$\|x_{n_0} - q\| < \frac{\varepsilon}{4M}. \tag{17}$$

From (15) to (17), for all $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - q\| + \|x_n - q\| \\ &\leq M \|x_{n_0} - q\| + \sum_{i=n_0}^{\infty} s_i + M \|x_{n_0} - q\| + \sum_{i=n_0}^{\infty} s_i \\ &< M \tfrac{\varepsilon}{4M} + \tfrac{\varepsilon}{4} + M \tfrac{\varepsilon}{4M} + \tfrac{\varepsilon}{4} = \varepsilon. \end{aligned} \tag{18}$$

This shows that $\{x_n\}$ is a Cauchy sequence, hence $x_n \to q \in K$. It remains to show that $q \in \mathcal{F}$. Notice that

$$|d(q, \mathcal{F}) - d(x_n, \mathcal{F})| \leq \|q - x_n\|, \qquad \forall n. \tag{19}$$

Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$,we obtain that $q \in \mathcal{F}$.
Since an asymptotically quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive mapping, so we have the following result.

Corollary 1. Let E be a Banach space and K a nonempty closed and convex subset of E and $\{T_i : i = 1, 2, \ldots, k\}$ a finite family of asymptotically quasi-nonexpansive self-mappings of K with the sequences $\{r_{ni}\}, \{l_{ni}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_{ni} < \infty$ and $\sum_{n=1}^{\infty} l_{ni} < \infty$ for all $i = 1, 2, \ldots, k$. Assume that $\mathcal{F} \neq \varnothing$ is closed and $\sum_{n=1}^{\infty} \gamma_{ni} < \infty$ for each $i = 1, 2, \ldots, k$. Then the iterative sequence $\{x_n\}, \{y_n\}, \ldots, \{y_{n+k-2}\}$ defined by (5) converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Theorem 2. Let E be a Banach space and K a nonempty closed and convex subset of E and $\{T_i : i = 1, 2, \ldots, k\}$ a finite family of generalized asymptotically quasi-nonexpansive self-mappings of K with the sequences $\{r_{ni}\}, \{l_{ni}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} r_{ni} < \infty$ and $\sum_{n=1}^{\infty} l_{ni} < \infty$ for all $i = 1, 2, \ldots, k$. Suppose that $\mathcal{F} \neq \varnothing$ is closed. Let $x_1 \in K$ and $\{x_n\}$ be the sequence defined by (5). If $\sum_{n=1}^{\infty} \gamma_{ni} < \infty, \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, k$ and $\{T_i : i = 1, 2, \ldots, k\}$ satisfies *Condition* $(\overline{C})$, then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

Proof. From $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, k$ and $\{T_i : i = 1, 2, \ldots, k\}$ satisfying *Condition* $(\overline{C})$, there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x_n - T_{i_0} x_n\| \geq f(d(x_n, \mathcal{F}))$ for some $i_0 \in \{1, 2, \ldots, k\}$, it follows that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. From Theorem 1, we obtain that $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

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