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On the Determinant of Tridiagonal Matrices via Some Special Numbers

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Abstract. In this study, we obtain the Generalized k-Fibonacci and k-Lucas numbers by using determinants of tridiagonal matrices. Therefore it has been established a new generalization for the tridiagonal matrices that represent well known *numbers* such as Fibonacci, Lucas, Pell and Pell-Lucas.

Key words: Generalized k-Fibonacci and k-Lucas numbers; tridiagonal matrix.
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1. Introduction

There is a huge interest of modern science in the application of Fibonacci and Lucas numbers. For instance, the ratio of two consecutive of these numbers converges to the Golden section $\alpha = \frac{1+\sqrt{5}}{2}$. In fact, it is well known that there is a direct relationship between Golden section and the numbers that interested in [7-9]. In the last years, in [1-4], there are the some generalizations of the Fibonacci and Lucas sequences. For instance, in [1,2], it has been defined k-Fibonacci $\{F_{k,n}\}_{n\in\mathbb{N}}$ and k-Lucas $\{L_{k,n}\}_{n\in\mathbb{N}}$ sequences by the recursive equations

$$(1) \qquad F_{k,n+2} = kF_{k,n+1} + F_{k,n} \quad \text{and} \quad L_{k,n+2} = kL_{k,n+1} + L_{k,n}, \ \text{for } k \in \mathbb{R}^+$$

with initial conditions $F_{k,0} = 0,\ F_{k,1} = 1$ and $L_{k,0} = 2,\ L_{k,1} = k$, respectively. After that, in [3] Authors considered the generalized k-Fibonacci and k-Lucas sequence $\{G_{k,n}\}_{n\in\mathbb{N}}$, which was defined by the recursive equation

$$(2) \qquad G_{k,n+2} = kG_{k,n+1} + G_{k,n}, \quad G_{k,0} = a,\ G_{k,1} = b$$

where $a, b \in \mathbb{Z}$, $k \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Considering [3], one can clearly obtain the characteristic equation of (2) as the form $t^2 - kt - 1 = 0$ with having the roots $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta = \frac{k-\sqrt{k^2+4}}{2}$. Note that

$$\alpha + \beta = k,\ \alpha\beta = -1 \text{ and } \alpha - \beta = \sqrt{k^2+4}. \tag{3}$$

Hence the Binet formula

$$G_{k,n} = \frac{X\ \alpha^n - Y\ \beta^n}{\alpha - \beta}, \tag{4}$$

where $X = \frac{a+b\alpha}{\alpha}$, $Y = \frac{a+b\beta}{\beta}$, can be thought as a solution of the recursive equation in (2).

As particular cases of this formula in (4):

· If $a = 0$ and $b = 1$, then we obtain Binet Formula for k-Fibonacci numbers in [1] as $F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$,

· If $a = 2$ and $b = k$, then we obtain Binet Formla for k-Lucas numbers in [2] as $L_{k,n} = \alpha^n + \beta^n$.

In these papers, especially, it has also been established the connection between Fibonacci and Lucas numbers in terms of $n \times n$ tridiagonal matrices $M(n)$. Moreover, concerning again $M(n)$, Strang [10, 11] interested a family of $M(n)$ which defined by

$$\begin{cases} m_{i,i} = 3; & 1 \le i \le n, \\ m_{i,i+1} = m_{i+1,i} = 1; & 1 \le i < n \end{cases}.$$

In these above references, for $1 \le k \le n$, it was presented that the determinant of each $M(k)$ is equal to the Fibonacci number F_{2k+2}. As another example, authors ([5, 6]) defined a new family of $M(n)$ which was given by

$$\begin{cases} m_{i,i} = 1; & 1 \le i \le n, \\ m_{j+1,j} = m_{j,,j+1} = i; & 1 \le i < n \end{cases}.$$

Similarly as above, for $2 \le k \le n$, it was also showed that the determinant of each $M(k)$ is equal to the Fibonacci number F_{k+1}.
The following proposition will be needed for one of our main result.

Proposition 1. [4] Let $A(k)$ be a family of tridiagonal matrices $M(k)$ defined by

$$A(k) = \begin{bmatrix} a_{1,1} & a_{1,2} & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & \\ & a_{3,2} & a_{3,3} & \ddots & \\ & & \ddots & \ddots & a_{k-1,k} \\ & & & a_{k,k-1} & a_{k,k} \end{bmatrix}.$$

Then the determinant of each $A(k)$ can be described by the following recurrence relations:

$\cdot\ |A(1)| = a_{1,1}$

$\cdot\ |A(2)| = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$

$\vdots$

$\cdot\ |A(k)| = a_{k,k}\det(A(k-1)) - a_{k,k-1}a_{k-1,k}\det(A(k-2))$, for $k \geq 3$.

Under these circumstances, as a new generalization of the studies on special numbers such as k-Fibonacci, k-Lucas, Fibonacci, Lucas, Pell and Pell-Lucas, in this paper we obtained the generalized k-Fibonacci and k-Lucas numbers in the meaning of determinants of $M(n)$. This will imply that all the previous results are actually as special cases of our results in this paper.

2. Main Result

Let us first consider the following lemma which can be proved easily by induction process. In fact, by this lemma, it will be given a new formula for the generalized k-Fibonacci and k-Lucas sequence under different indices.

Lemma 1. For $n \geq 1$, we have

$$G_{k,n+2i} = L_{k,i}G_{k,n+i} - (-1)^i G_{k,n}. \tag{5}$$

Proof. From the Binet Formulas for k-Lucas and the generalized k-Fibonacci and k-Lucas numbers, we have

$$L_{k,i}G_{k,n+i} - (-1)^i G_{k,n} = \left(\alpha^i + \beta^i\right)\left(\frac{X\alpha^{n+i} - Y\beta^{n+i}}{\alpha - \beta}\right) - (-1)^i\left(\frac{X\alpha^n - Y\beta^n}{\alpha - \beta}\right).$$

By considering (3), if we rewrite this last equality, then we get

$$\begin{aligned} L_{k,i}G_{k,n+i} - (-1)^i G_{k,n} &= \frac{X\alpha^{n+2i} - Y\beta^{n+2i}}{\alpha - \beta} \\ &= G_{k,n+2i}, \end{aligned}$$

as required.

In the rest of this paper, the tridiagonal matrices

$$\begin{bmatrix} G_{k,r+s} & \sqrt{m_{2,2}G_{k,r+s} - G_{k,2r+s}} & & & \\ \sqrt{m_{2,2}G_{k,r+s} - G_{k,2r+s}} & \left\lceil \frac{G_{k,2r+s}}{G_{k,r+s}} \right\rceil & \sqrt{(-1)^r} & & \\ & \sqrt{(-1)^r} & L_{k,r} & \ddots & \\ & & \ddots & \ddots & \sqrt{(-1)^r} \\ & & & \sqrt{(-1)^r} & L_{k,r} \end{bmatrix}$$

will be denoted by $M_{r,s}(n)$, where $r, s \in \mathbb{Z}$, instead of $M(n)$. The elements of $M_{r,s}(n)$ will be defined in detail in Theorem 1 and Corollary 1 below.
The following main result establishes that the generalized k-Fibonacci and k-Lucas numbers can be obtained as a linear combination of the indices of $\det(M_{r,s}(n))$.

Theorem 1. For $1 \leq p \leq n$, the elements of $M_{r,s}(p)$ are

$$m_{1,1} = G_{k,r+s}, \quad m_{2,2} = \left\lceil \frac{G_{k,2r+s}}{G_{k,r+s}} \right\rceil, \ \cdots, \ m_{j,j} = L_{k,r} \quad (3 \leq j \leq n),$$

$$m_{1,2} = m_{2,1} = \sqrt{m_{2,2}G_{k,r+s} - G_{k,2r+s}}, \cdots,$$

$$m_{j,j+1} = m_{j+1,j} = \sqrt{(-1)^r} \quad (2 \leq j < n),$$

where $r \in \mathbb{Z}^+$ and $s \in \mathbb{N}$.
In addition, we have $\det(M_{r,s}(p)) = G_{k,rp+s}$.

Proof. Let us use the principle of mathematical induction on p.
While, for $p = 1$,

$$\det(M_{r,s}(1)) = G_{k,r+s},$$

it is easy to see that, for $p = 2$,

$$\det(M_{r,s}(2)) = G_{k,2r+s}.$$

As the usual next step of induction, let us assume that it is true for all positive integers p. That is,

$$\det(M_{r,s}(p)) = G_{k,rp+s}.$$

Therefore, we have to show that it is true for $p + 1$. In other words, we need to check

$$\det(M_{r,s}(p+1)) = G_{k,r(p+1)+s}.$$

Considering Proposition 1, we have

$$\det(M_{r,s}(p+1)) = L_{k,r}G_{k,rp+s} - (-1)^r G_{k,r(p-1)+s}.$$

Also, by Lemma 1, we obtain

$$\det(M_{r,s}(p+1)) = G_{k,r(p+1)+s}$$

which ends up the induction and the proof.

Corollary 1. In above Theorem, if we take $r = s = 1$, then the matrix becomes

$$M_{1,1}(n) = \begin{bmatrix} G_{k,2} & \sqrt{m_{2,2}G_{k,2} - G_{k,3}} & & & \\ \sqrt{m_{2,2}G_{k,2} - G_{k,3}} & \left\lceil \frac{G_{k,3}}{G_{k,2}} \right\rceil & i & & \\ & i & k & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & k \end{bmatrix}.$$

Then $det(M_{1,1}(n)) = G_{k,n+1}$.

Particular cases of previous Corollary are:

· For $a = 0,\ b = 1$, it is obtained k-Fibonacci numbers $F_{k,n}$. In here $det(M_{1,1}(n)) = F_{k,n+1}$,

· For $a = 2,\ b = k$, it is obtained k-Lucas numbers $L_{k,n}$.In here $det(M_{1,1}(n)) = L_{k,n+1}$,

· By choosing suitable values on a, b and k, some special numbers can also be obtained in terms of $det(M_{1,1}(n))$.

After all, by a different approximation, we can also define the generalized k-Fibonacci and k-Lucas numbers with the determinant of the tridiagonal matrix

$$M(n) = \begin{bmatrix} b & a & & & \\ -1 & k & 1 & & \\ & -1 & k & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & k \end{bmatrix}. \tag{6}$$

For the matrix $M(n)$ in (6), we get $det(M(n)) = G_{k,n}$. Moreover, for special choices of $a = 1, b = 2,\ k = 2$, matrix in (6) becomes the matrix

$$\begin{bmatrix} 2 & 1 & & & \\ -1 & 2 & 1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & 2 \end{bmatrix}$$

which the determinant of it is actually equal to the Pell number P_{n+1}.

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