Selçuk J. Appl. Math.
Vol. 13. No. 1. pp. 3-9, 2012

Selçuk Journal of Applied Mathematics

Approximate Solution of the Double Nonlinear Singular Integral Equations with Hilbert Kernel by the Method of Contractive Mappings

Nushaba F. Gasimova

Ministry of Education of Azerbaijan Republic, Khatai(Xatai) prospect, 49, AZ 1008, Azerbaijan
e-mail: enusabe@yahoo.com

Received Date: September 7, 2010
Accepted Date: March 8, 2011

Abstract. In this paper the double nonlinear singular integral equations with Hilbert kernel are solved by contractive mappings method and the rate of convergence of sequential approximations to exact solution is found.

Key words: Approximate solution; Singular integral equations; Bicylindrical domain; Superposition; Cfontractive mappings
2000 Mathematics Subject Classification: 45G05.

1. Introduction

Some notations and auxiliary facts
Let's consider the following double nonlinear singular integral equation (NSIE) of the form

$$\varphi(x,y)=\lambda\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}F[s,t,\varphi(s,t)]ctg\frac{s-x}{2}ctg\frac{t-y}{2}\;ds\;dt+f(x,y), \tag{1.1}$$

where λ is a real parameter, F and f are the given functions, φ is the desired function. Equations of the form (1.1) are met by studying limit values on the frames of bicylinder of the function which is analytic in bicylindrical domain [1] and the theory of singular integral equations [2]. In this paper we'll solve equation (1.1) by the contractive mappings method.
By $C(T^2)$ we denote a space of continuous functions on $T^2=[-\pi,\pi]\times[-\pi,\pi]$ and have 2π periodic by each of variables with the norm

$$\|f\|_{C(T^2)}=\max_{(x,y)\in T^2}|f(x,y)|. \tag{1.2}$$

Let

$$\triangle_h^{1,0} f(x,y) = f(x+h,y) - f(x,y), \quad \triangle_\eta^{0,1} f(x,y) = f(x,y+\eta) - f(x,y),$$

$$\triangle_{h,\eta}^{1,1} f(x,y) = f(x,y) - f(x+h,y) - f(x,y+\eta) + f(x+h,y+\eta). \tag{1.3}$$

These quantities are called partial difference with respect to x with step h, with respect to c with step η and mixed difference in aggregate of variables with step h and η at the point (x,y).
Introduce the denotation:

$$\omega_f^{1,0}(\delta) = \sup_{|h|\le\delta} \|\triangle_h^{1,0} f(x,y)\|_{C(T^2)}, \quad \omega_f^{0,1}(\eta) = \sup_{|h|\le\eta} \|\triangle_h^{0,1} f(x,y)\|_{C(T^2)},$$

$$\omega_f^{1,1}(\delta,\eta) = \sup_{\substack{|h_1|\le\delta \\ |h_2|\le\eta}} \|\triangle_{h_1,h_2}^{1,1} f(x,y)\|_{C(T^2)}. \tag{1.4}$$

By means of these characteristics in the paper [3] we introduce the space
(1.5)

$$K_{\alpha,\beta}^{1,1} = \left\{ f \in C(T^2) \,\middle|\, \omega_f^{1,0}(\delta) = O(\delta^\alpha),\ \omega_f^{0,1}(\eta) = O(\eta^\beta),\ \omega_f^{1,1}(\delta,\eta) = O(\delta^\alpha \cdot \eta^\beta) \right\} \quad 0 < \alpha, \beta \le 1$$

with finite norm

$$\|f\|_{K_{\alpha,\beta}^{1,1}} = \max\left\{ \|f\|_{C(T^2)},\ \sup_{\delta>0} \frac{\omega_f^{1,0}(\delta)}{\delta^\alpha},\ \sup_{\eta>0} \frac{\omega_f^{0,1}(\eta)}{\eta^\beta},\ \sup_{\substack{\delta>0 \\ \eta>0}} \frac{\omega_f^{1,1}(\delta)}{\delta^\alpha \cdot \eta^\beta} \right\}$$

and prove that the spaces $K_{\alpha,\beta}^{1,1}$ is a Banach space.
Let $f \in C(T^2)$. Let's consider a double singular integral with Hilbert kernel

$$\tilde{f}(x,y) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s,t) ctg \frac{s-x}{2} ctg \frac{t-y}{2}\ ds\ dt. \tag{1.6}$$

Note that integral (1.6) is understood in the sense of Cauchy's principal value. From the estimates obtained in the papers [3], [4] it follows that the singular operator (SO)

$$(Sf)(x,y) = \tilde{f}(x,y) \tag{1.7}$$

Acts from $K_{\alpha,\beta}^{1,1}$ to $K_{\alpha,\beta}^{1,1}$ and bounded for $0 < \alpha,\ \beta < 1$.

In the space $K^{1,1}_{\alpha,\beta}$ we take a ball with center at zero of radius R

$$B^{1,1}_{\alpha,\beta}(R)=\left\{\varphi\in K^{1,1}_{\alpha,\beta}\mid \|\varphi\|_{K^{1,1}_{\alpha,\beta}}\le R\right\}.$$

The following statement was proved in the paper [5].

Statement 1. Let $f\in K^{1,1}_{\alpha,\beta}$ and $1\le p<\infty$. Then the inequality

$$\|f\|_{C(T^2)}\le l\|f\|^{\gamma}_{K^{1,1}_{\alpha,\beta}}\cdot\|f\|^{1-\gamma}_{L_p}, \tag{1.8}$$

where $\gamma=\dfrac{1+p(\alpha+\beta)}{(1+\alpha p)(1+\beta p)}$,

$$l=\max\left\{\frac{(1+\alpha p)(1+\beta p)}{(\alpha\beta p)^{1-\gamma}},\frac{\sqrt[p]{4}(1+\alpha p)(1+\beta p)}{\alpha\beta p\pi^{1-\gamma}}\right\} \tag{1.9}$$

is true.
Later on we'll need the following statements proved in [10].

Statement 2. Let the function $F(x,y,\varphi):T^2\times[-R,R]\to\Re$ satisfy the conditions:

1) there exists a partial derivative $F'_\varphi(x,y,\varphi)$ and there is $C_0>0$ such that for $\forall\varphi_1,\varphi_2\in[-R,R]$ $|F'_\varphi(x,y,\varphi_1)-F'_\varphi(x,y,\varphi_2)|\le C_0|\varphi_1-\varphi_2|$;
2) $\exists C_1>0$, $\forall x_1,x_2\in[-\pi,\pi]$ $|F(x_1,y,\varphi)-F(x_2,y,\varphi)|\le C_1|x_1-x_2|^\alpha$;
3) $\exists C_2>0$, $\forall y_1,y_2\in[-\pi,\pi]$ $|F(x,y_1,\varphi)-F(x,y_2,\varphi)|\le C_2|y_1-y_2|^\beta$;
4) $\exists C_3>0$, $\forall x_1,y_1,x_2,y_2\in[-\pi,\pi]$
$|F(x_1,y_1,\varphi)-F(x_1,y_2,\varphi)-F(x_2,y_1,\varphi)+F(x_2,y_2,\varphi)|\le C_3|x_1-x_2|^\alpha|y_1-y_2|^\beta$;
5) $\exists C_4>0$, $\forall x_1,x_2\in[-\pi,\pi]$, $\forall\varphi_1,\varphi_2\in[-R,R]$
$|F(x_1,y,\varphi_1)-F(x_1,y,\varphi_2)-F(x_2,y,\varphi_1)+F(x_2,y,\varphi_2)|\le C_4|x_1-x_2|^\alpha|\varphi_1-\varphi_2|$;
6) $\exists C_5>0$, $\forall y_1,y_2\in[-\pi,\pi]$, $\forall\varphi_1,\varphi_2\in[-R,R]$
$|F(x,y_1,\varphi_1)-F(x,y_1,\varphi_2)-F(x,y_2,\varphi_1)+F(x,y_2,\varphi_2)|\le C_5|y_1-y_2|^\beta|\varphi_1-\varphi_2|$.

Then the operator of superposition $F:\varphi(x,y)\to F[x,y,\varphi(x,y)]$ acts from the ball $B^{1,1}_{\alpha,\beta}(R)$ to the ball $B^{1,1}_{\alpha,\beta}(R_1)$ where radius R_1 is uniquely determined by initial data.

Statement 3. Let the function $F(s,t,\varphi):T^2\times[-R,R]\to\Re$ satisfy conditions 1)- 6) and $f\in B^{1,1}_{\alpha,\beta}(R')$ $(R'<R)$.
Then for

$$\lambda<\min\left\{\frac{1}{C^*\|S\|_{L_2\to L_2}},\frac{R-R'}{R_1\cdot\|S\|_{K^{1,1}_{\alpha,\beta}\to K^{1,1}_{\alpha,\beta}}}\right\}, \tag{1.10}$$

where $C^*=\max\limits_{x,y,\varphi}|F'_\varphi(x,y,\varphi)|$, the operator

$$(L\varphi)(x,y)=\lambda\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}F[s,t,\varphi(s,t)]ctg\frac{s-x}{2}ctg\frac{t-y}{2}\,ds\,dt+f(x,y) \tag{1.11}$$

Is a contractive map in the ball $B^{1,1}_{\alpha,\beta}(R)$ in the metric of the space $L_2(T^2)$.

2. Approximate Solution of NSIE (1.1)

From estimate (1.8) it follows that if a sequence of functions $\{f_n\} \subset B^{1,1}_{\alpha,\beta}(R)$ converges in the metric of the space $L_2(T^2)$ to some function f_0, it converges to f_0 in the metric of the space $C(T^2)$ as well.

It is valid.

Lemma 2.1. *If the sequence* $\{f_u\} \subset B^{1,1}_{\alpha,\beta}(R)$ converges in the metric of space $C\left(T^2\right)$ to f_0, then $f_0 \in B^{1,1}_{\alpha,\beta}(R)$.

Proof. $f_n \to f_0$ $f_n \in B^{1,1}_{\alpha,\beta}(R)$. Then

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \quad \forall n > N(\varepsilon),\ \forall (x,y) \in T^2 |f_n(x,y) - f(x,y)| < \varepsilon. \tag{2.1}$$

Let's take arbitrary points $(x_1, y), (x_2, y) \in T^2$ and arbitrary $\varepsilon_0 > 0$ and fix them. Take such $\varepsilon > 0$ that the inequality $\dfrac{\varepsilon}{|x_1 - x_2|^\alpha} < \varepsilon_0$ be fulfilled. Then we have

$$\begin{aligned} &\tfrac{|f_0(x_1,y)-f_0(x_2,y)|}{|x_1-x_2|^\alpha} = \tfrac{|f_0(x_1,y)-f_n(x_1,y)+f_n(x_1,y)-f_n(x_2,y)+f_n(x_2,y)-f_0(x_2,y)|}{|x_1-x_2|^\alpha} \\ &\le \tfrac{|f_0(x_1,y)-f_n(x_1,y)|}{|x_1-x_2|^\alpha} + \tfrac{|f_n(x_1,y)-f_n(x_2,y)|}{|x_1-x_2|^\alpha} + \tfrac{|f_n(x_2,y)-f_0(x_2,y)|}{|x_1-x_2|^\alpha} < 2\varepsilon_0 + R. \end{aligned} \tag{2.2}$$

The relation

$$\frac{f_0(x, y_1) - f_0(x, y_2)}{|y_1 - y_2|^\beta} < 2\varepsilon_0 + R \tag{2.3}$$

is proved similarly. Now, let's fix the points $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2) \in T^2$ and $\varepsilon_0 > 0$. Take such $\varepsilon > 0$ that the relation $\dfrac{\varepsilon}{|x_1 - x_2|^\alpha |y_1 - y_2|^\beta} < \varepsilon_0$ be fulfilled.

Then

$$\begin{aligned} \tfrac{|f_0(x_1,y_1)-f_0(x_1,y_2)-f_0(x_2,y_1)+f_0(x_2,y_2)|}{|x_1-x_2|^\alpha|y_1-y_2|^\beta} &= (|f_0(x_1,y_1) - f_n(x_1,y_1) \\ &+ f_n(x_1,y_1) - f_0(x_1,y_2) + f_n(x_1,y_2) - f_n(x_1,y_2) \\ &- f_0(x_2,y_1) + f_n(x_2,y_1) - f_n(x_2,y_1) + f_0(x_2,y_2) \\ &- f_n(x_2,y_2) + f_n(x_2,y_2)|) \ / \ |x_1 - x_2|^\alpha |y_1 - y_2|^\beta \\ \le \tfrac{|f_0(x_1,y_1)-f_n(x_1,y_1)|}{|x_1-x_2|^\alpha|y_1-y_2|^\beta} &+ \tfrac{|f_0(x_1,y_2)-f_n(x_1,y_2)|}{|x_1-x_2|^\alpha|y_1-y_2|^\beta} + \tfrac{|f_0(x_2,y_1)-f_n(x_2,y_1)|}{|x_1-x_2|^\alpha|y_1-y_2|^\beta} \\ + \tfrac{|f_0(x_2,y_2)-f_n(x_2,y_2)|}{|x_1-x_2|^\alpha|y_1-y_2|^\beta} &+ \tfrac{|f_n(x_1,y_1)-f_n(x_1,y_2)-f_n(x_2,y_1)+f_n(x_2,y_2)|}{|x_1-x_2|^\alpha|y_1-y_2|^\beta} \\ &< 4\varepsilon_0 + R \end{aligned} \tag{2.4}$$

It follows from estimates (2.2)-(2.4) that $f_0 \in B^{1,1}_{\alpha,\beta}(R+4\varepsilon)$. Since $\varepsilon_0 > 0$ is arbitrary, we get $f_0 \in B^{1,1}_{\alpha,\beta}(R)$. The lemma is proved.

Now, let's prove the main theorem:

Theorem 2.1. *Let the function $F(s,t,\varphi) : T^2 \times [-R,R] \to \Re$ satisfy conditions 1) - 6) and $f \in B^{1,1}_{\alpha,\beta}(R')$ $(R' < R)$. Then for*

$$|\lambda| < \min\left\{ \frac{1}{C^* \|S\|_{L_2(T^2)}}, \frac{R-R'}{R_1 \|S\|_{K^{1,1}_{\alpha,\beta} \to K^{1,1}_{\alpha,\beta}}} \right\}$$

NSIE (1.1) has a unique solution φ^* in the ball $B^{1,1}_{\alpha,\beta}(R)$ and sequential approximations $\varphi_n = L\varphi_{n-1}$ converge to this solution in the metric $C(T^2)$ with rate

$$\|\varphi_n - \varphi^*\|_{C(T^2)} \le M \cdot \omega^n \|\varphi_1 - \varphi_0\|^{\frac{\gamma}{1+\gamma}}_{L_2(T^2)},$$

where M is a constant,

$$\omega = \{|\lambda| C^* \|S\|_{L_2(T^2)}\}^{\frac{\gamma}{1+\gamma}}, \quad \gamma = \min\{\alpha, \beta\}.$$

Proof. Under the conditions of the theorem $L$is contractive map in the metric $L_2(T^2)$. Then by contractive mappings principle we get

(2.6)

$$\|\varphi_n - \varphi^*\|_{L_2(T2)} \le \frac{1}{1-\omega_0} \omega_0^n \|\varphi_1 - \varphi_0\|_{L_2(T^2)}, \text{ where } \omega_0 = |\lambda| C^* \|S\|_{L_2(T^2)}$$

Estimate the norm $\|\varphi_n - \varphi^*\|_{C_2(T^2)}$ by the norm $\|\varphi_n - \varphi^*\|_{L_2(T^2)}$. By $B((x,y);h)$ we denote a circle of radius $h > 0$ and center at the point $(x,y) \in T^2$. Later on, let $V_2 = V_2^h(x,y) = T^2 \cap B((x,y);h)$. It is clear that for the function $g \in C(T^2)$ it holds the representation [9]:

$$g(x,y) = \frac{1}{mesV_2} \int_{V_2}\!\!\int g(s,t)\, ds\, dt - \frac{1}{mesV_2} \int_{V_2}\!\!\int [g(s,t) - g(x,y)]\, ds\, dt$$

Having taken $g(x,y) = \varphi^*(x,y) - \varphi_n(x,y)$ we get:

$$\varphi^*(x,y) - \varphi_n(x,y) = \frac{1}{mesV_2} \int_{V_2}\!\!\int [\varphi^*(s,t) - \varphi_n(s,t)] ds\, dt \tag{2.7}$$
$$- \frac{1}{mesV_2} \int_{V_2}\!\!\int [\varphi^*(s,t) - \varphi^*(x,y) - \varphi_n(s,t) + \varphi_n(x,y)] ds\, dt$$

Since $\varphi^*\varphi_n \in B^{1,1}_{\alpha,\beta}(R)$, we have

$$
\begin{aligned}
&|\varphi^*(s,t) - \varphi^*(x,y) - \varphi_n(s,t) + \varphi_n(x,y)| \le \\
&\qquad \le |\varphi^*(s,t) - \varphi^*(s,y) - \varphi^*(x,t) + \varphi^*(x,y)| \\
&\qquad\qquad + |\varphi^*(s,y) - \varphi^*(x,y)| + |\varphi^*(x,t) - \varphi^*(x,y)| \\
&\qquad\qquad - \ |\varphi_n(s,t) + \varphi_n(s,y) - \varphi_n(x,t) + \varphi_n(x,y)| \\
&\qquad\qquad + \ |\varphi_n(s,y) - \varphi_n(x,y)| + |\varphi_n(x,y) - \varphi_n(x,y)| \\
&\qquad \le 2M_1(|s-x|^\alpha |t-y|^\beta + |s-x|^\alpha + |t-y|^\beta) \le 2M_2 h^\gamma.
\end{aligned}
$$

And here M_2 is a constant and $\gamma = \min\{\alpha, \beta\}$. Then it follows from (2.7) that

$$
(2.8) \qquad |\varphi^*(x,y) - \varphi_n(x,y)| \le \frac{1}{\sqrt{mesV_2}} A_n + M_2 h^\gamma \le M_3 h^{-1} A_n + M_2 h^\gamma,
$$

where $An = \|\varphi_n - \varphi^*\|_{L_2(T^2)}$.
If we take $h = A_n^{\frac{1}{1+\gamma}}$, we have from (2.8)

$$
\begin{aligned}
&|\varphi^*(x,y) - \varphi_n(x,y)| \le M_4 A_n^{\frac{\gamma}{1+\gamma}} = M_4 \|\varphi_n - \varphi^*\|_{L_2(T^2)}^{\frac{\gamma}{1+\gamma}} \Rightarrow \\
&\qquad \Rightarrow |\varphi^* - \varphi_n|_{C(T^2)} \le M_4 \|\varphi_n - \varphi^*\|_{L_2(T^2)}^{\frac{\gamma}{1+\gamma}}.
\end{aligned}
$$

Taking into account the last inequality and taking $M = M_4 \left(\frac{1}{1-\omega_0}\right)^{\frac{\gamma}{1+\gamma}}$ we get the affirmation of the theorem.
The theorem is proved.

References

1. Kakichev V.A., Schwartz integral and Hilbert formula for analytical functions of many complex variables, Izv. Vuzov. matem., No 2, (1959), 80-93.
2. Gabdulkhaev B.G., Approximate solutions of multidimensional singular integral equations, II. Izv. Vuzov. matem., No 1, (1976), 30-41.
3. Musayev B.I., Salayev V.V., On conjugated functions of many groups of variables, II. Uchen.zap. Azerb. Univ. Ser. Fiz. Mat., No 4, (1978), 5-18.
4. Jvarsheishvili A.G., On A.Zigmund's inequality for functions of two variables, Soobsh.AN. Gruz. SSR., v.15, No 9, (1954), 9-15.
5. Abdullayev F.A., On some multiplicative inequalities for functions of two variables, Azerb. Resp. "Tahsil" , No 2, (2004), 56-60 (Russian).
6. Zhizhiashvili L.V., On some problems from the theory of prime and multiple trigonometric and orthogonal series, Uspekhi mat. Nauk, v.28, No 2, (1973), 65-119.
7. Babayev A.A., On structure of a linear operator and its application, Uchen.Zap.AGU, ser.fiz.-mat.nauk, No 4, (1961), 13-16.
8. Huseynov A.I., Mukhtarov Kh.Sh., Introduction to the theory of nonlinear singular integral equations, "Nauk" (1979).

9. Il'in V.P. Some inequalities in functional spaces and their application to research of convergence of variational processes, Trudy Inst. Matem. AN SSSR. v.53, (1959), 64-121.
10. Gasimova N.F. On a nonlinear singular integral operator with Hilbert kernel, Baku Universitetinin Khabarleri. Fiz.mat.ser., No 4, (2005), 56-63.