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On the Periods of Some Figurate Numbers

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Abstract. The number which can be represented by a regular geometrical arrangement of equally spaced point is defined as figurate number. Each of polygonal, centered polygonal and pyramidal numbers is a class of the series of figurate numbers. In this paper, we obtain the periods of polygonal, centered polygonal and pyramidal numbers by reducing each element of these numbers modulo m.

Key words: Period, polygonal number, centered polygonal number, pyramidal number.
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1. Introduction

The figurate numbers have a very important role to solve some problems in number theory and to determine speciality of some numbers, see for example, [8,9]. The polygonal numbers, the centered polygonal numbers, the pyramidal numbers and their properties have been studied by some authors, see for example, [1,3,6,15]. The study of Fibonacci numbers by reducing modulo m began with the earlier work of Wall [13] where the periods of Fibonacci numbers according to modulo m were obtained. The theory is expanded to 3-step Fibonacci sequence by Özkan, Aydin and Dikici [11]. Lü and Wang [10] contributed to study of the Wall number for the $k-$step Fibonacci sequence. Deveci and Karaduman [5] extended the concept to Pell numbers. Now we extend the concept to the polygonal numbers, the centered polygonal numbers and the pyramidal numbers which are classes of the series of figurate numbers.

A sequence is *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the *period of the sequence.* For example, the sequence $a, b, c, b, c, b, c, \cdots$ is periodic after the initial element a and has period 2. A sequence is *simply periodic* with period n if the first n elements in the sequence form a repeating subsequence. For example, the sequence $a, b, c, a, b, c, a, b, c, , \cdots$ is simply periodic with period 3.
We have the following formulas for the polygonal numbers, the centered polygonal numbers and the pyramidal numbers:
Let k be the number of sides in a polygon. The n^{th} $k-$gonal number is obtained by the formula

$$P(k,n) = (\frac{k}{2} - 1)n^2 - \left(\frac{k}{2} - 2\right) n.$$

A polygonal number is a number represented as dots or pebbles arranged in the shape of a regular polygon. We obtain n^{th} triangular number, n^{th} square number, n^{th} pentagonal number, $\cdots$ for $k = 3, 4, 5, \cdots$. For more information on $k-$gonal numbers, see [12].
The n^{th} centered $k-$gonal number is obtained by the formula

$$C(k,n) = \frac{kn}{2}(n-1) + 1.$$

A centered polygonal number formed by a central dot, surrounded by polygonal layers with a constant number of sides. Each side of a polygonal layer contains one dot more than a side in the previous layer, so starting from the second polygonal layer each layer of a centered $k-$gonal number contains k more points than the previous layer. We obtain n^{th} centered triangular number, n^{th} centered square number, n^{th} centered pentagonal number, $\cdots$ for $k = 3, 4, 5, \cdots$. For more information on centered $k-$gonal numbers, see [2].
The $n^{th}k-$gonal pyramidal number is obtained by formula

$$P_n^{(k)} = \frac{n^2}{2} + n^3 \left(\frac{k}{6} - \frac{1}{3}\right) - n\left(\frac{k-5}{6}\right).$$

A pyramidal number represents a pyramid with a base and given number of sides. We obtain n^{th} triangular pyramidal number, n^{th} square pyramidal number, n^{th} pentagonal pyramidal number, $\cdots$ for $k = 3, 4, 5, \cdots$. For more information on $k-$gonal pyramidal numbers, see [14].
In this paper, the usual notation p is used for a prime number.

2. Polygonal Numbers

In this section, we obtain the lengths of the periods of $k-$gonal numbers modulo m. The notation $L_P(m)$ denote the length of the smallest period which the period is obtained each element of the polygonal numbers by reducing modulo m for $k \geq 3$.

Theorem 2.1. Let $k \equiv 0 \pmod 4$. The lengths of the periods of the polygonal numbers are as follows:

i. For $u \in \mathbb{N}$, $L_P(2^u) = \begin{cases} 2 & \text{for } u = 1, \\ 2^{u-1} & \text{for } u > 2. \end{cases}$

ii. If $p \neq 2$ and $\theta \in \mathbb{N}$, then $L_P(p^\theta) = p^\theta$.

iii. If $m = \prod_{i=1}^{t} p_i^{e_i}$ $(t \geq 1)$ where p_i's are distinct primes, then $L_P(m) = \text{lcm}\left[L_P\left(p_i^{e_i}\right)\right]$.

Proof. We prove this by direct calculation. Since $k \equiv 0 \pmod 4$ and $k \geq 3$, $k = 4\ell$, $(\ell \in \mathbb{N})$.

i. Since

$P(4\ell, n) \pmod 2 \equiv \left[(2\ell-1)n^2 - (2\ell-2)n\right] \pmod 2 \equiv 0 \pmod 2$ for n is even and

$P(4\ell, n) \pmod 2 \equiv \left[(2\ell-1)n^2 - (2\ell-2)n\right] \pmod 2 \equiv 1 \pmod 2$ for n is odd, $L_P(2) = 2$. If $u > 1$, then

$$\begin{aligned}
& P\left(4\ell, 2^{u-1}\right) \pmod{2^u} \equiv \left[(2\ell-1)\left(2^{u-1}\right)^2 + (2\ell-2)\,2^{u-1}\right] \pmod{2^u} \equiv \\
& \equiv \left[2^u\ell\left(2^{u-1}-1\right) + 2^u\left(1-2^{u-2}\right)\right] \pmod{2^u} \equiv 0 \pmod{2^u}, \\
& P\left(4\ell, 2^{u-1}+1\right) \pmod{2^u} \equiv \\
& \equiv \left[(2\ell-1)\left(2^{u-1}+1\right)^2 + (2\ell-2)\left(2^{u-1}+1\right)\right] \pmod{2^u} \equiv \\
& \equiv \left[2^u\left(\ell + 2^{u-1} - 2^{u-2}\right) + 1\right] \pmod{2^u} \equiv 1 \pmod{2^u}, \cdots, \\
& P\left(4\ell, 2^{u-1}+n\right) \pmod{2^u} \equiv \\
& \equiv \left[(2\ell-1)\left(2^{u-1}+n\right)^2 + (2\ell-2)\left(2^{u-1}+n\right)\right] \pmod{2^u} \equiv \\
& \equiv \left[2^u\left(2^{u-1}\ell + 2\ell n - 2^{u-2} - n - \ell + 1\right) + 1\right] \pmod{2^u} + \\
& + \left[(2\ell-1)n^2 + (2\ell-2)n\right] (\text{mod}\, d\ 2^u) \equiv \\
& \equiv \left[(2\ell-1)n^2 + (2\ell-2)n\right] \pmod{2^u} \equiv P(4\ell, n) \pmod{2^u}.
\end{aligned}$$

So, we get $L_P(2^u) = 2^{u-1}$.

ii. The proof is similar to the proof of i. and is omitted.

iii. Let $\text{lcm}\left[L_P\left(p_i^{e_i}\right)\right] = \beta$.

$$\begin{aligned}
& P(4\ell, \beta) \pmod m \equiv \left[(2\ell-1)\beta^2 + (2\ell-2)\beta\right] \pmod m \equiv 0 \pmod m, \\
& P(4\ell, \beta+1) \pmod m \equiv \left[(2\ell-1)(\beta+1)^2 + (2\ell-2)(\beta+1)\right] \pmod m \equiv \\
& \equiv \left[\beta^2(2\ell-1) + 1\right] \pmod m \equiv 1 \pmod m, \cdots, \\
& P(4\ell, \beta) \pmod m \equiv \left[(2\ell-1)\beta^2 + (2\ell-2)\beta\right] \pmod m \equiv 0 \pmod m, \\
& P(4\ell, \beta+1) \pmod m \equiv \left[(2\ell-1)(\beta+1)^2 + (2\ell-2)(\beta+1)\right] \pmod m \equiv \\
& \equiv \left[\beta^2(2\ell-1) + 1\right] \pmod m \equiv 1 \pmod m, \cdots,
\end{aligned}$$

$$P(4\ell, \beta+n) \pmod{m} \equiv \left[(2\ell-1)(\beta+n)^2 + (2\ell-2)(\beta+n)\right] \pmod{m} \equiv$$
$$\equiv [\beta(2\ell\beta + 4\ell n - \beta - 2n - 2\ell + 2) + 1] \pmod{m} +$$
$$+\left[(2\ell-1)n^2 + (2\ell-2)n\right] \pmod{m} \equiv$$
$$\equiv \left[(2\ell-1)n^2 + (2\ell-2)n\right] \pmod{m} \equiv P(4\ell, n) \pmod{m}.$$

So, we get $L_P(m) = \text{lcm}[L_P(p_i^{e_i})]$.

Theorem 2.2: Let $k \equiv 2 \pmod{4}$. Then $L_P(m) = m$ for $m \geq 2$.

Proof: We prove this by direct calculation. Since $k \equiv 2 \pmod{4}$ and $k \geq 3$, $k = 4\ell + 2,\ (\ell \in \mathbb{N})$.

$$P(4\ell+2, m+n) \pmod{m} \equiv \left[(2\ell)(m+n)^2 + (2\ell-1)(m+n)\right] \pmod{m} \equiv$$
$$\equiv m(2\ell m + 2n - 2\ell - 1) \pmod{m} + \left[(2\ell)(n)^2 + (2\ell-1)(n)\right] \pmod{m} \equiv$$
$$\equiv P(4\ell+2, n) \pmod{m}.$$

So, we get $L_P(m) = m$.

Theorem 2.3. Let $k \equiv 1 \pmod{4}$ or $k \equiv 3 \pmod{4}$. The lengths of the periods of the polygonal numbers are as follows:
i. For $u \in \mathbb{N}$, $L_P(2^u) = 2^{u+1}$.
ii. If $p \neq 2$ and $\theta \in \mathbb{N}$, then $L_P(p^\theta) = p^\theta$.
iii. If $m = \prod_{i=1}^{t} p_i^{e_i}\ (t \geq 1)$ where p_i's are distinct primes, then $L_P(m) = \text{lcm}[L_P(p_i^{e_i})]$.

Proof. The proof is similar to the proof of Theorem 2.1. and is omitted.

3. Centered Polygonal Numbers

In this section, we obtain the lengths of the periods of centered $k-$gonal numbers modulo m. The notation $L_{CP}(m)$ denote the length of the smallest period which the period is obtained each element of the centered polygonal numbers by reducing modulo m for $k \geq 3$.

Theorem 3.1. Let $k \equiv 0 \pmod{4}$ that is $k = 2^\alpha.p_1^{e_1}.p_2^{e_2}.\cdots.p_t^{e_t}$, $\alpha \geq 2$ such that $p_1, p_2, \cdots, p_t$ are distinct primes and $e_1, e_2, \cdots, e_t$ are 0 or positive integers. The lengths of the periods of the centered polygonal numbers are as follows:
i. For $u \in \mathbb{N}$, $L_{CP}(2^u) = \begin{cases} 1 & \text{for } u \leq \alpha, \\ 2^{u-1} & \text{for } u > \alpha. \end{cases}$
ii. If $v \in \mathbb{N}$ such that $1 \leq v \leq t$ and $\lambda \in \mathbb{N}$, then $L_{CP}(p_v^\lambda) = \begin{cases} 1 & \text{for } \lambda \leq e_v, \\ p_v^{\lambda-e_v} & \text{for } \lambda > e_v. \end{cases}$
iii. If $p \neq 2, p_1, p_2, \cdots, p_t$ and $\theta \in \mathbb{N}$, then $L_{CP}(p^\theta) = p^\theta$.

Proof. We prove this by direct calculation. Since $k \equiv 0 \pmod 4$ and $k \geq 3$, $k = 2^{\alpha}.p_1^{e_1}.p_2^{e_2}.\cdots.p_t^{e_t}$, $\alpha \geq 2$ such that $p_1, p_2, \cdots, p_t$ are distinct primes and $e_1, e_2, \cdots, e_t$ are 0 or positive integers.
i. If $u \leq \alpha$, then

$$C(k,n) \pmod{2^u} \equiv \left[\frac{k}{2}n(n-1)+1\right] \pmod{2^u} \equiv 1 \pmod{2^u}.$$

So, we get $L_{CP}(2^u) = 1$. If $u > \alpha$, then

$$\begin{aligned}
&C\left(k,2^{u-1}\right) \pmod{2^u} \equiv \left[\tfrac{k}{2}2^{u-1}\left(2^{u-1}-1\right)+1\right] \pmod{2^u} \equiv 1 \pmod{2^u},\\
&C\left(k,2^{u-1}+1\right) \pmod{2^u} \equiv \left[\tfrac{k}{2}\left(2^{u-1}+1\right)2^{u-1}+1\right] \pmod{2^u} \equiv 1 \pmod{2^u}, \cdots,\\
&C\left(k,2^{u-1}+n\right) \pmod{2^u} \equiv \left[\tfrac{k}{2}\left(2^{u-1}+n\right)\left(2^{u-1}+n-1\right)+1\right] \pmod{2^u} \equiv\\
&\equiv \left[\tfrac{k}{2}\left(2^u n + 2^{u-1}\left(2^{u-1}-1\right)\right)\right] \pmod{2^u} + \left[\tfrac{k}{2}n(n-1)+1\right] \pmod{2^u} \equiv\\
&\equiv \left[\tfrac{k}{2}n(n-1)+1\right] \pmod{2^u} \equiv C(k,n) \pmod{2^u}.
\end{aligned}$$

So, we get $L_{CP}(2^u) = 2^{u-1}$.
ii. If $\lambda \leq e_v$ and $1 \leq v \leq t$, then

$$C(k,n) \left(\bmod\ p_v^{\lambda}\right) \equiv \left[\frac{k}{2}n(n-1)+1\right] \left(\bmod\ p_v^{\lambda}\right) \equiv 1 \left(\bmod\ p_v^{\lambda}\right).$$

So, we get $L_{CP}\left(p_v^{\lambda}\right) = 1$. If $\lambda > e_v$ and $1 \leq v \leq t$, then

$$\begin{aligned}
&C\left(k,p_v^{\lambda-e_v}\right) \left(\bmod\ p_v^{\lambda}\right) \equiv \left[\tfrac{k}{2}p_v^{\lambda-e_v}\left(p_v^{\lambda-e_v}-1\right)+1\right] \left(\bmod\ p_v^{\lambda}\right) \equiv 1 \left(\bmod\ p_v^{\lambda}\right),\\
&C\left(k,p_v^{\lambda-e_v}+1\right) \left(\bmod\ p_v^{\lambda}\right) \equiv\\
&\equiv \left[\tfrac{k}{2}\left(p_v^{\lambda-e_v}+1\right)p_v^{\lambda-e_v}+1\right] \left(\bmod\ p_v^{\lambda}\right) \equiv 1 \left(\bmod\ p_v^{\lambda}\right), \cdots,
\end{aligned}$$

$$\begin{aligned}
&C\left(k,p_v^{\lambda-e_v}+n\right) \left(\bmod\ p_v^{\lambda}\right) \equiv\\
&\equiv \left[\tfrac{k}{2}\left(p_v^{\lambda-e_v}+n\right)\left(p_v^{\lambda-e_v}+n-1\right)+1\right] \left(\bmod\ p_v^{\lambda}\right) \equiv\\
&\equiv \left[\tfrac{k}{2}\left(2np_v^{\lambda-e_v}+p_v^{\lambda-e_v}\left(p_v^{\lambda-e_v}-1\right)\right)\right] \left(\bmod\ p_v^{\lambda}\right) +\\
&+\left[\tfrac{k}{2}n(n-1)+1\right] \left(\bmod\ p_v^{\lambda}\right) \equiv\\
&\equiv \left[\tfrac{k}{2}n(n-1)+1\right] \left(\bmod\ p_v^{\lambda}\right) \equiv C(k,n) \left(\bmod\ p_v^{\lambda}\right).
\end{aligned}$$

So, we get $L_{CP}\left(p_v^{\lambda}\right) = p_v^{\lambda-e_v}$.
iii. If $p \neq 2, p_1, p_2, \cdots, p_t$, then

$$\begin{aligned}
&C\left(k,p^{\theta}\right) \left(\bmod\ p^{\theta}\right) \equiv \left[\tfrac{k}{2}p^{\theta}\left(p^{\theta}-1\right)+1\right] \left(\bmod\ p^{\theta}\right) \equiv 1 \left(\bmod\ p^{\theta}\right),\\
&C\left(k,p^{\theta}+1\right) \left(\bmod\ p^{\theta}\right) \equiv \left[\tfrac{k}{2}\left(p^{\theta}+1\right)p^{\theta}+1\right] \left(\bmod\ p^{\theta}\right) \equiv 1 \left(\bmod\ p^{\theta}\right), \cdots,\\
&C\left(k,p^{\theta}+n\right) \left(\bmod\ p^{\theta}\right) \equiv \left[\tfrac{k}{2}\left(p^{\theta}+n\right)\left(p^{\theta}+n-1\right)+1\right] \left(\bmod\ p^{\theta}\right) \equiv\\
&\equiv \left[\tfrac{k}{2}\left(2np^{\theta}+p^{\theta}\left(p^{\theta}-1\right)\right)\right] \left(\bmod\ p^{\theta}\right) + \left[\tfrac{k}{2}n(n-1)+1\right] \left(\bmod\ p^{\theta}\right) \equiv\\
&\equiv \left[\tfrac{k}{2}n(n-1)+1\right] \left(\bmod\ p_v^{\lambda}p^{\theta}\right) \equiv C(k,n) \left(\bmod\ p^{\theta}\right).
\end{aligned}$$

So, we get $L_{CP}\left(p^{\theta}\right)=p^{\theta}$.

Theorem 3.2. Let $k\equiv 2 \pmod 4$ that is $k=2.p_1^{e_1}.p_2^{e_2}.\cdots.p_t^{e_t}$ such that $p_1,p_2,\cdots,p_t$ are distinct primes and $e_1,e_2,\cdots,e_t$ are positive integers. The lengths of the periods of the centered polygonal numbers are as follows:

i. For $u\in\mathbb{N}$, $L_{CP}\left(2^u\right)=\begin{cases}1 & \text{for } u=1,\\ 2^u & \text{for } u>1.\end{cases}$

ii. If $v\in\mathbb{N}$ such that $1\le v\le t$ and $\lambda\in\mathbb{N}$, then $L_{CP}\left(p_v^{\lambda}\right)=\begin{cases}1 & \text{for } \lambda\le e_v,\\ p_v^{\lambda-e_v} & \text{for } \lambda>e_v.\end{cases}$

iii. If $p\ne 2,p_1,p_2,\cdots,p_t$ and $\theta\in\mathbb{N}$, then $L_{CP}\left(p^{\theta}\right)=p^{\theta}$.

Proof. We prove this by direct calculation. Since $k\equiv 2 \pmod 4$ and $k\ge 3$, $k=2.p_1^{e_1}.p_2^{e_2}.\cdots.p_t^{e_t}$ such that $p_1,p_2,\cdots,p_t$ are distinct primes and $e_1,e_2,\cdots,e_t$ are positive integers.
i. $C(k,n) \pmod 2\equiv\left[\frac{k}{2}n(n-1)+1\right] \pmod 2\equiv 1 \pmod 2$.
So, we get $L_{CP}(2)=1$. If $u>1$, then

$$\begin{aligned}
&C\left(k,2^u\right) \pmod{2^u}\equiv\left[\tfrac{k}{2}2^u\left(2^u-1\right)+1\right] \pmod{2^u}\equiv 1 \pmod{2^u},\\
&C\left(k,2^u+1\right) \pmod{2^u}\equiv\left[\tfrac{k}{2}\left(2^u+1\right)2^u+1\right] \pmod{2^u}\equiv 1 \pmod{2^u},\cdots,\\
&C\left(k,2^u+n\right) \pmod{2^u}\equiv\left[\tfrac{k}{2}\left(2^u+n\right)\left(2^u+n-1\right)+1\right] \pmod{2^u}\equiv\\
&\equiv\left[\tfrac{k}{2}\left(2^{u+1}n+2^u\left(2^u-1\right)\right)\right] \pmod{2^u}+\left[\tfrac{k}{2}n(n-1)+1\right] \pmod{2^u}\equiv\\
&\equiv\left[\tfrac{k}{2}n(n-1)+1\right] \pmod{2^u}\equiv C(k,n) \pmod{2^u}.
\end{aligned}$$

So, we get $L_{CP}\left(2^u\right)=2^u$.
The proofs of ii. and iii. are similar to the proofs of Theorem 3.1. ii. and Theorem 3.1.iii., respectively and are omitted.

Theorem 3.3. Let $k\equiv 3 \pmod 4$ that is $k=p_1^{e_1}.p_2^{e_2}.\cdots.p_t^{e_t}$ such that $p_1,p_2,\cdots,p_t$ are distinct primes and $e_1,e_2,\cdots,e_t$ are positive integers. The lengths of the periods of the centered polygonal numbers are as follows:

i. For $u\in\mathbb{N}$, $L_{CP}\left(2^u\right)=2^{u+1}$.

ii. If $v\in\mathbb{N}$ such that $1\le v\le t$ and $\lambda\in\mathbb{N}$, then $L_{CP}\left(p_v^{\lambda}\right)=\begin{cases}1 & \text{for } \lambda\le e_v,\\ p_v^{\lambda-e_v} & \text{for } \lambda>e_v.\end{cases}$

iii. If $p\ne 2,p_1,p_2,\cdots,p_t$ and $\theta\in\mathbb{N}$, then $L_{CP}\left(p^{\theta}\right)=p^{\theta}$.

Proof. We prove this by direct calculation. Since $k\equiv 3 \pmod 4$, $k=p_1^{e_1}.p_2^{e_2}.\cdots.p_t^{e_t}$ such that $p_1,p_2,\cdots,p_t$ are distinct primes and $e_1,e_2,\cdots,e_t$ are positive integers.

$$\begin{aligned}
&C\left(k,2^{u+1}\right) \pmod{2^u}\equiv\left[\tfrac{k}{2}2^{u+1}\left(2^{u+1}-1\right)+1\right] \pmod{2^u}\equiv 1 \pmod{2^u},\\
&C\left(k,2^{u+1}+1\right) \pmod{2^u}\equiv\left[\tfrac{k}{2}\left(2^{u+1}+1\right)2^{u+1}+1\right] \pmod{2^u}\equiv 1 \pmod{2^u},\cdots,\\
&C\left(k,2^{u+1}+n\right) \pmod{2^u}\equiv\left[\tfrac{k}{2}\left(2^{u+1}+n\right)\left(2^{u+1}+n-1\right)+1\right] \pmod{2^u}\equiv\\
&\equiv\left[\tfrac{k}{2}\left(2^{u+2}n+2^{u+1}\left(2^{u+1}-1\right)\right)\right] \pmod{2^u}+\left[\tfrac{k}{2}n(n-1)+1\right] \pmod{2^u}\equiv\\
&\equiv\left[\tfrac{k}{2}n(n-1)+1\right] \pmod{2^u}\equiv C(k,n) \pmod{2^u}.
\end{aligned}$$

So, we get $L_{CP}(2^u) = 2^{u+1}$.
The proofs of ii. and iii. are similar to the proofs of Theorem 3.1. ii. and Theorem 3.1.iii., respectively and are omitted.
If $k \equiv 1 \pmod 4$, then the rules are the same the rules of Theorem 3.3. The proof is similar to the proof of Theorem 3.3 and is omitted.

Theorem 3.4. If $m = \prod_{i=1}^{t} p_i^{e_i}$ $(t \geq 1)$ where p_i's are distinct primes, then $L_{CP}(m) = \mathrm{lcm}\,[L_{CP}(p_i^{e_i})]$.

Proof. We prove this by direct calculation. Let $\mathrm{lcm}\,[L_{CP}(p_i^{e_i})] = \beta$.

$$C(k,\beta) \pmod m \equiv \left[\tfrac{k}{2}\beta(\beta-1)+1\right] \pmod m \equiv 1 \,(\mathrm{mod}\ m),$$
$$C(k,\beta+1) \pmod m \equiv \left[\tfrac{k}{2}(\beta+1)\beta+1\right] \pmod m \equiv 1 \,(\mathrm{mod}\ m), \cdots,$$
$$C(k,\beta+n) \pmod m \equiv \left[\tfrac{k}{2}(\beta+n)(\beta+n-1)+1\right] \pmod m \equiv$$
$$\equiv \left[\tfrac{k}{2}\beta(\beta+2n-1)\right] \pmod m + \left[\tfrac{k}{2}n(n-1)+1\right] \pmod m \equiv$$
$$\equiv \left[\tfrac{k}{2}n(n-1)+1\right] \pmod m \equiv C(k,n) \pmod{2^u}.$$

So, we get $L_{CP}(m) = \mathrm{lcm}\,[L_{CP}(p_i^{e_i})]$.

4. Pyramidal Numbers

In this section, we obtain the lengths of the periods of $k-$gonal pyramidal numbers modulo m. The notation $L_{PY}(m)$ denote the length of the smallest period which the period is obtained each element of the pyramidal numbers by reducing modulo m for $k \geq 3$.

Theorem 4.1. Let $k \equiv 0 \pmod 3$ or $k \equiv 1 \pmod 3$. The lengths of the periods of the pyramidal numbers are as follows:
i. If $p = 2, 3$, then $L_{PY}(p^u) = p^{u+1}$ for $u \in \mathbb{N}$.
ii. If $p \neq 2, 3$ and $\theta \in \mathbb{N}$, then $L_{PY}(p^\theta) = p^\theta$.
iii. If $m = \prod_{i=1}^{t} p_i^{e_i}$ $(t \geq 1)$ where p_i's are distinct primes, then $L_{PY}(m) = \mathrm{lcm}\,[L_{PY}(p_i^{e_i})]$.

Proof. We prove this by direct calculation. Let $k \equiv 0 \pmod 3$ that is $k = 3\ell$, $(\ell \in \mathbb{N})$.
i. If $p = 2$, then we have for $u \in \mathbb{N}$

$$P^{(3\ell)}_{2^{u+1}-1} \pmod{2^u} \equiv$$
$$\equiv \left[\frac{(2^{u+1}-1)^2}{2} + (2^{u+1}-1)^3\left(\tfrac{3\ell}{6}-\tfrac{1}{3}\right) - (2^{u+1}-1)\left(\tfrac{3\ell-5}{6}\right)\right] \pmod{2^u} \equiv$$
$$\equiv \left[\tfrac{2^u}{3}\left(5-2^{2u+3}\right)\right] \pmod{2^u} \equiv 0 \,(\mathrm{mod}\ 2^u),$$
$$P^{(3\ell)}_{2^{u+1}} \pmod{2^u} \equiv \left[\frac{(2^{u+1})^2}{2} + (2^{u+1})^3\left(\tfrac{3\ell}{6}-\tfrac{1}{3}\right) - (2^{u+1})\left(\tfrac{3\ell-5}{6}\right)\right] \pmod{2^u} \equiv$$
$$\equiv \left[\tfrac{2^u}{3}\left(5-2^{2u+3}\right)\right] \pmod{2^u} \equiv 0 \,(\mathrm{mod}\ 2^u),$$

$$P^{(3\ell)}_{2^{u+1}+1} \pmod{2^u} \equiv$$
$$\equiv \left[\frac{\left(2^{u+1}+1\right)^2}{2} + \left(2^{u+1}+1\right)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right) - \left(2^{u+1}+1\right)\left(\frac{3\ell-5}{6}\right)\right] \pmod{2^u} \equiv$$
$$\equiv \left[\frac{2^u}{3}\left(5-2^{2u+3}\right)\right] + 1 \pmod{2^u} \equiv 1 \pmod{2^u}, \cdots,$$
$$P^{(3\ell)}_{2^{u+1}+n} \pmod{2^u} \equiv$$
$$\equiv \left[\frac{\left(2^{u+1}+n\right)^2}{2} + \left(2^{u+1}+n\right)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right) - \left(2^{u+1}+n\right)\left(\frac{3\ell-5}{6}\right)\right] \pmod{2^u} \equiv$$
$$\equiv \left[\frac{2^u}{3}\left(5-2^{2u+3}\right) \pmod{2^u}\right] + \left[\frac{n^2}{2} + n^3\left(\frac{3\ell}{6}-\frac{1}{3}\right) - n\left(\frac{3\ell-5}{6}\right)\right] \pmod{2^u} \equiv$$
$$\equiv P^{(3\ell)}_n \pmod{2^u}.$$

So, we get $L_{PY}\left(2^u\right) = 2^{u+1}$.
If $p = 3$, then the proof is similar to the proof of the case $p = 2$ and is omitted.
ii. If $p \neq 2, 3$ and $\theta \in \mathbb{N}$, then we have

$$P^{(3\ell)}_{p^u-1} \pmod{p^u} \equiv$$
$$\equiv \left[\frac{(p^u-1)^2}{2} + (p^u-1)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right) - (p^u-1)\left(\frac{3\ell-5}{6}\right)\right] \pmod{p^u} \equiv$$
$$\equiv \left[\frac{p^u}{6}(5-2.p^u)(p^u+1)\right] \pmod{p^u} \equiv 0 \pmod{p^u},$$
$$P^{(3\ell)}_{p^u} \pmod{p^u} \equiv \left[\frac{(p^u)^2}{2} + (p^u)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right) - (p^u)\left(\frac{3\ell-5}{6}\right)\right] \pmod{p^u} \equiv$$
$$\equiv \left[\frac{p^u}{6}(5-2.p^u)(p^u+1)\right] \pmod{p^u} \equiv 0 \pmod{p^u},$$
$$P^{(3\ell)}_{p^u+1} \pmod{p^u} \equiv$$
$$\equiv \left[\frac{(p^u+1)^2}{2} + (p^u+1)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right) - (p^u+1)\left(\frac{3\ell-5}{6}\right)\right] \pmod{p^u} \equiv$$
$$\equiv \left[\frac{p^u}{6}(5-2.p^u)(p^u+1)+1\right] \pmod{p^u} \equiv 1 \pmod{p^u}, \cdots,$$
$$P^{(3\ell)}_{p^u+n} \pmod{p^u} \equiv$$
$$\equiv \left[\frac{(p^u+n)^2}{2} + (p^u+n)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right) - (p^u+n)\left(\frac{3\ell-5}{6}\right)\right] \pmod{p^u} \equiv$$
$$\equiv \left[\frac{p^u}{6}(5-2.p^u)(p^u+1)\right] \pmod{p^u} +$$
$$+\left[\frac{n^2}{2} + n^3\left(\frac{3\ell}{6}-\frac{1}{3}\right) - n\left(\frac{3\ell-5}{6}\right)\right] \pmod{2^u} \equiv P^{(3\ell)}_n \pmod{2^u}.$$

So, we get $L_{PY}\left(p^\theta\right) = p^\theta$.

iii. Let $\operatorname{lcm}\left[L_{PY}\left(p_i^{e_i}\right)\right]=\beta$.

$P_{\beta-1}^{(3\ell)}\pmod{m}\equiv\left[\frac{(\beta-1)^2}{2}+(\beta-1)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right)-(\beta-1)\left(\frac{3\ell-5}{6}\right)\right]\pmod{m}\equiv$

$\equiv\left[\frac{\beta}{6}(5-2.\beta)(\beta+1)\right]\pmod{m}\equiv 0\pmod{m},$

$P_{\beta}^{(3\ell)}\pmod{m}\equiv\left[\frac{(\beta)^2}{2}+(\beta)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right)-(\beta)\left(\frac{3\ell-5}{6}\right)\right]\pmod{m}\equiv$

$\equiv\left[\frac{\beta}{6}(5-2.\beta)(\beta+1)\right]\pmod{m}\equiv 0\pmod{m},$

$P_{\beta+1}^{(3\ell)}\pmod{m}\equiv\left[\frac{(\beta+1)^2}{2}+(\beta+1)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right)-(\beta+1)\left(\frac{3\ell-5}{6}\right)\right]\pmod{m}\equiv$

$\equiv\left[\frac{\beta}{6}(5-2.\beta)(\beta+1)+1\right]\pmod{m}\equiv 1\pmod{m},\cdots,$

$P_{\beta+n}^{(3\ell)}\pmod{m}\equiv\left[\frac{(\beta+n)^2}{2}+(\beta+n)^3\left(\frac{3\ell}{6}-\frac{1}{3}\right)-(\beta+n)\left(\frac{3\ell-5}{6}\right)\right]\pmod{m}\equiv$

$\left[\frac{\beta}{6}(5-2.\beta)(\beta+1)\right]\pmod{m}+\left[\frac{n^2}{2}+n^3\left(\frac{3\ell}{6}-\frac{1}{3}\right)-n\left(\frac{3\ell-5}{6}\right)\right]\pmod{m}\equiv$

$\equiv P_n^{(3\ell)}\pmod{m}.$

So, we get $L_{PY}(m)=\operatorname{lcm}\left[L_{PY}\left(p_i^{e_i}\right)\right]$.

Theorem 4.2. Let $k\equiv 2\pmod 3$. The lengths of the periods of the pyramidal numbers are as follows:
i. For $u\in\mathbb{N}$, $L_{PY}(2^u)=2^{u+1}$.
ii. If $p\neq 2$ and $\theta\in\mathbb{N}$, then $L_{PY}\left(p^\theta\right)=p^\theta$.
iii. If $m=\prod_{i=1}^{t}p_i^{e_i}$ $(t\geq 1)$ where p_i's are distinct primes, then $L_{PY}(m)=\operatorname{lcm}\left[L_{PY}\left(p_i^{e_i}\right)\right]$.

Proof. The proof is similar to the proof of Theorem 4.1. and is omitted.

5. Open Question

Wall in [13] proved that the lengths of the periods of the recurring sequences obtained by reducing a Fibonacci sequences by a modulo m are equal to the lengths of the of ordinary $2-$step Fibonacci recurrences in cyclic groups. The theory is expanded to 3-step Fibonacci sequence by Özkan, Aydin and Dikici [11]. Lü and Wang contributed to the study of the Wall number for the k-step Fibonacci sequence [10]. Some works on the concept have been made, for example, [4,7]. Deveci and Karaduman [5] extended the concept to Pell sequences in finite groups. Are there groups such that the lengths of the periods of recurrence sequences obtained by reducing according to a modulo m anyone of polygonal numbers, centered polygonal numbers and pyrimidal numbers are equal to the lengths of the periods of $2-$step or $k-$step Fibonacci recurrences in these groups?

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