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The Solution of the Nonlinear Dispersive K(m,n,1) Equations by RDT Method

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Abstract. In the present paper, we implement the Reduced Differential Transform Method to solve the nonlinear dispersive K(m,n,1) type equations. This method is an alternative approach which is capable of reducing significantly the size of calculations unlike the classical differential transformation to overcome relatively troublesome aspects of perturbation techniques and the Adomian decomposition method regarding computational simplicity. To illustrate the applicability of the proposed method, two special types K(2,2,1) and K(3,3,1) of dispersive equations are discussed. Numerical results have been found in good agreement with the exact solutions.

Key words: Reduced Differential Transform Method, Nonlinear dispersive equations, Adomian Decomposition Method, Variational Iteration Method.
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1.Introduction

Searching for the solitary solutions to nonlinear equations plays an important role in soliton theory. For example, compactons can be described as solitons with finite wave length or solitons that don't have exponential tails and they are a new class of localized solitons for the families of nonlinear dispersive partial differential equations. There are many examples of nonlinear equations such as Korteweg-de Vries (KdV) equation, mKdV equation, RLW equation, Sine-Gordon equation, Boussinesq equation and Burgers' equation, etc., applicable in engineering, fluid mechanics, biology, mathematics and physics (for example, plasma physics and solid state physics). Lots of recent studies have focused their attentions on the theory of nonlinear problems mentioned above. Wadati developed solutions to KdV and mKdV equations in [1-3]. Here, we will mention a simple form of the well known KdV equation:

(1) $$u_t - auu_x + u_{xxx} = 0.$$

The dispersion term u_{xxx} in equation (1) makes the wave form spread. Rosenau and Hyman [4] presented a class of compactons of nonlinear equations:

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3 \tag{2}$$

which is called fully nonlinear dispersive $K(m,n)$ equations. Solitons and compactons are studied by many approximation techniques such as Adomian decomposition method [5-7], homotopy perturbation method [8-11], variational iteration method [12-16], He's semi inverse method [17], Differential Transform Method [18], Multi-step Differential Transform method[19] and Exp function method [20-22], etc.
In this paper, we will apply the semi-functional or reduced differential transform method (RDTM) [23,24] to solve the nonlinear dispersive $K(m,n,1)$ type equations:

$$u_t + (u^m)_x - (u^n)_{xxx} + u_{5x} = 0, \quad m > 1, \quad 1 \leq n \leq 3, \tag{3}$$

with the initial condition

$$u(x,0) = f(x). \tag{4}$$

In particular, the proposed method is discussed for two special types of $K(m,n,1)$ equations. It is also noted that throughout the paper, all calculations are executed in Maple package programming environment.

2. Analysis of the Method

This method is first proposed by Keskin and Oturanc in [23]. The basic definitions of Reduced Differential Transform Method [23-25] are introduced as follows:

Definition 2.1. Let the function $u(x,t)$ is analytic and continuously differentiable with respect to time t and space x in the domain of the interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}, \tag{5}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represents the original function while the uppercase $U_k(x)$ stands for the transformed function.

Definition 2.2. The differential inverse transform of $U_k(x)$is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k. \tag{6}$$

Then combining equations (5) and (6), we write

$$u(x,t) = \sum_{k=0}^{n} \frac{1}{k!} \left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} t^k. \tag{7}$$

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion.
For the purpose of illustration of the methodology by the proposed method, we write the nonlinear dispersive $K(m,n,1)$ equation in the standard operator form

$$L(u(x,t)) + R(u(x,t)) + N(u(x,t)) = g(x,t), \tag{8}$$

with the initial condition

$$u(x,0) = f(x), \tag{9}$$

where $L = \frac{\partial}{\partial t}$ is a linear operator, $N(u(x,t)) = (u^m)_x - (u^n)_{xxx}$ is a nonlinear term, $R(u(x,t)) = u_{5x}$ is a remaining linear term and $g(x,t)$ is a homogeneous term. Some basic operations of the RDTM are given in Table 1 that shows the procedure of a Maple code for the nonlinear part of eq. (8) in its last row.
According to the table 1, we can develop the following iteration formula:

$$(k+1)U_{k+1}(x) = G_k(x) - R(U_k(x)) - N(U_k(x)), \tag{10}$$

where $R(U_k(x))$, $N(U_k(x))$ and $G_k(x)$ are the transformations of the functions $R(u(x,t))$, $N(u(x,t))$ and $g(x,t)$ respectively. We can write the first few nonlinear terms as

$$N_0 = \left(\frac{\partial}{\partial x} U_0^m(x) - \frac{\partial^3}{\partial x^3} U_0^n(x)\right),$$
$$N_1 = \left(\frac{\partial}{\partial x} m U_0^{m-1}(x) U_1(x) - \frac{\partial^3}{\partial x^3} n U_0^{n-1}(x) U_1(x)\right),$$
$$N_2 = \begin{pmatrix} \frac{\partial}{\partial x}\left(m(m-1)U_0^{m-2}(x)U_1(x) + mU_0^{m-1}(x)U_2(x)\right) \\ -\frac{\partial^3}{\partial x^3}\left(n(n-1)U_0^{n-2}(x)U_1(x) + nU_0^{n-1}(x)U_2(x)\right) \end{pmatrix}.$$

The transformation of the initial condition (9) gives

$$U_0(x) = f(x), \tag{11}$$

Substituting (11) into (10) and after recursive calculations, we get the coefficients $U_k(x)\ (k = 1, 2, \ldots)$. Then, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{\infty}$ gives an approximate solution as,

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^k + \Re_{n+1}(x,t), \tag{12}$$

where

$$\Re_{n+1}(x,t) = \sum_{k=n+1}^{\infty} U_k(x) t^k$$

is called a remainder and n shows the order of the approximation. Therefore it is possible to get the exact solution of the problem by

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t), \tag{13}$$

Table 1. Operations of reduced differential transformation

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α is a constant)
$w(x,y) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
$w(x,y) = x^m t^n u(x,t)$	$W_k(x) = x^m U(k-n)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)...(k+r)U_{k+1}(x) = \frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{\partial}{\partial x}U_k(x)$
$Nu(x,t)$	**Maple Code for** *Nonlinear Function* *restart;* *NF:=Nu(x,t):#Nonlinear Function* *m:=5: # Order* *u[t]:=sum(u[b]*t^b,b=0..m):* *NF[t]:=subs(Nu(x,t)=u[t],NF):* *s:=expand(NF[t],t):* *dt:=unapply(s,t):* *for i from 0 to m do* *n[i]:=((D@@i)(dt)(0)/i!):* *print(N[i],n[i]);* #Transform Function *od:*

3. Applications

In this section, two examples $K(2,2,1)$ and $K(3,3,1)$of nonlinear dispersive equations are chosen to illustrate the procedure of the RDTM. The results are compared with the Adomian solutions and those of the Variational iteration method to appreciate the efficiency and the effectiveness of the proposed scheme.

3.1. Example. Let us consider the nonlinear dispersive $K(2,2,1)$equation

$$u_t + (u^2)_x - (u^2)_{xxx} + u_{5x} = 0, \tag{14}$$

with the initial condition

$$u(x,0)=\frac{16c-1}{12}\cosh^2(\frac{x}{4}), \tag{15}$$

where c is an arbitrary constant. Applying the reduced differential transform to (14), we obtain the recurrence equation

$$(k+1)U_{k+1}(x)=\frac{\partial^3}{\partial x^3}N_k(x)-\frac{\partial}{\partial x}N_k(x)-\frac{\partial^5}{\partial x^5}U_k(x), \tag{16}$$

where $U_k(x)$ is the t-dimensional spectrum function of $u(x,t)$ and $N_k(x)$ is the transformation of the function $u^2(x,t)$. From the initial condition (15), we can write the initial transformation term

$$U_0(x)=\frac{16c-1}{12}\cosh^2(\frac{x}{4}). \tag{17}$$

Substituting the initial transformation (17) in (16), we get the coefficient$U_1(x)$. Therefore, successive substitutions of $U_k(x)\,(k=1,2,\ldots)$ in (16) give the required coefficients as

$$\begin{aligned}
U_0(x) &= \frac{16c-1}{12}\cosh^2(\frac{x}{4})\\
U_1(x) &= -\frac{1}{24}c(16c-1)\cosh(\frac{x}{4})\sinh(\frac{x}{4})\\
U_2(x) &= \frac{1}{192}c^2(16c-1)\left(2\cosh^2(\frac{x}{4})-1\right)\\
U_3(x) &= -\frac{1}{576}c^3(16c-1)\cosh(\frac{x}{4})\sinh(\frac{x}{4})\\
U_4(x) &= \frac{1}{9216}c^4(16c-1)\left(2\cosh^2(\frac{x}{4})-1\right)\\
U_5(x) &= -\frac{1}{46080}c^5(16c-1)\cosh(\frac{x}{4})\sinh(\frac{x}{4})\\
U_6(x) &= \frac{1}{1105920}c^6(16c-1)\left(2\cosh^2(\frac{x}{4})-1\right)\\
U_7(x) &= -\frac{1}{7741440}c^7(16c-1)\cosh(\frac{x}{4})\sinh(\frac{x}{4})\\
&\ \ \vdots
\end{aligned}$$

Then, using the inverse transformation, we get the approximated solution
(18)

$$\begin{aligned}
\tilde{u}(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k &=\tfrac{16c-1}{12}\cosh^2(\tfrac{x}{4})-\tfrac{1}{24}c(16c-1)\cosh(\tfrac{x}{4})\sinh(\tfrac{x}{4})t\\
&+\tfrac{1}{192}c^2(16c-1)\left(2\cosh^2(\tfrac{x}{4})-1\right)t^2-\tfrac{1}{576}c^3(16c-1)\cosh(\tfrac{x}{4})\sinh(\tfrac{x}{4})t^3\\
&+\tfrac{1}{9216}c^4(16c-1)\left(2\cosh^2(\tfrac{x}{4})-1\right)t^4-\tfrac{1}{46080}c^5(16c-1)\cosh(\tfrac{x}{4})\sinh(\tfrac{x}{4})t^5\\
&+\tfrac{1}{1105920}c^6(16c-1)\left(2\cosh^2(\tfrac{x}{4})-1\right)t^6-\tfrac{1}{7741440}c^7(16c-1)\cosh(\tfrac{x}{4})\sinh(\tfrac{x}{4})t^7+...
\end{aligned}$$

Eventually, it is easy to see the closed form solution of the above series as

$$u(x,t) = \frac{16c-1}{12}\cosh^2\left(\frac{ct-x}{4}\right), \tag{19}$$

which coincides with the exact solution of the problem in [6] and in [13]. For comparison reasons, the RDTM solution of order seven is plotted together with the exact solution in Fig.1.a and with the solution of three-term variational iteration method in Figure 1.b.

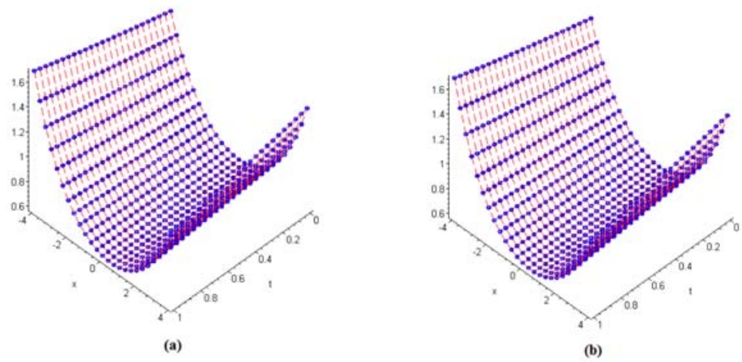

Figure 1. (a) Comparison of the solution of the equation K(2,2,1) by RDTM of order seven (solid line) with the exact solution (point) for c=0.5.
(b) Comparison of the solution of the equation K(2,2,1) by RDTM of order seven (solid line) with the VIM solution with three terms (point) for c=0.5.

3.2. Example. We, now, consider the nonlinear dispersive $K(3,3,1)$ equation

$$u_t + (u^3)_x - (u^3)_{xxx} + u_{5x} = 0, \tag{20}$$

with the initial condition

$$u(x,0) = \sqrt{\frac{81c-1}{54}}\cosh(\frac{x}{3}), \tag{21}$$

where c is an arbitrary constant. Applying the reduced differential transform to (20), we obtain the recurrence relation

$$(k+1)U_{k+1}(x) = \frac{\partial^3}{\partial x^3}N_k(x) - \frac{\partial}{\partial x}N_k(x) - \frac{\partial^5}{\partial x^5}U_k(x), \tag{22}$$

where $U_k(x)$ is the t-dimensional spectrum function of $u(x,t)$ and $N_k(x)$ is the transformation of the function $u^3(x,t)$. From the initial condition (21), we can write the initial transformation term

$$U_0(x) = \sqrt{\frac{81c-1}{54}}\cosh(\frac{x}{3}). \tag{23}$$

Substituting (23) into (22), we find the first term of the approximation. Then, following the successive substitutions in (22), we get the following $U_k(x)$(k=1,2,...) values

$$\begin{array}{ll} U_0(x) = \frac{1}{18}\sqrt{486c-6}\cosh(\frac{x}{3}) & U_1(x) = -\frac{1}{54}c\sqrt{486c-6}\sinh(\frac{x}{3}) \\ U_2(x) = \frac{1}{324}c^2\sqrt{486c-6}\cosh(\frac{x}{3}) & U_3(x) = -\frac{1}{2916}c^3\sqrt{486c-6}\sinh(\frac{x}{3}) \\ U_4(x) = \frac{1}{34992}c^4\sqrt{486c-6}\cosh(\frac{x}{3}) & U_5(x) = -\frac{1}{524880}c^5\sqrt{486c-6}\sinh(\frac{x}{3}) \\ U_6(x) = \frac{1}{9447840}c^6\sqrt{486c-6}\cosh(\frac{x}{3}) & U_7(x) = -\frac{1}{198404640}c^7\sqrt{486c-6}\sinh(\frac{x}{3}) \\ \vdots & \end{array}$$

Using the inverse transformation, we write the solution in a series form

$$\begin{aligned} \tilde{u}(x,t) = & \tfrac{1}{18}\sqrt{486c-6}\cosh(\tfrac{x}{3}) - \tfrac{1}{54}c\sqrt{486c-6}\sinh(\tfrac{x}{3})t \\ & + \tfrac{1}{324}c^2\sqrt{486c-6}\cosh(\tfrac{x}{3})t^2 - \tfrac{1}{2916}c^3\sqrt{486c-6}\sinh(\tfrac{x}{3})t^3 \\ & + \tfrac{1}{34992}c^4\sqrt{486c-6}\cosh(\tfrac{x}{3})t^4 - \tfrac{1}{524880}c^5\sqrt{486c-6}\sinh(\tfrac{x}{3})t^5 \\ & + \tfrac{1}{9447840}c^6\sqrt{486c-6}\cosh(\tfrac{x}{3})t^6 - \tfrac{1}{198404640}c^7\sqrt{486c-6}\sinh(\tfrac{x}{3})t^7 + \cdots \end{aligned} \tag{24}$$

Therefore, the exact solution of the problem can be given by

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t)$$

and from equation (24), it is easy to verify that the closed form solution can be written by

$$u(x,t) = \sqrt{\frac{81c-1}{54}\cosh\left(\frac{ct-x}{3}\right)} \tag{25}$$

which coincides with the exact solution of the problem in [6] and in [13]. Even exact solution of the problem is known, for comparison purposes, the graphical representation of the RDTM solution of order seven is shown in Figure 2.a together with the exact solution, and it is also compared with the solution of three-terms variational iteration method in Figure 2.b.

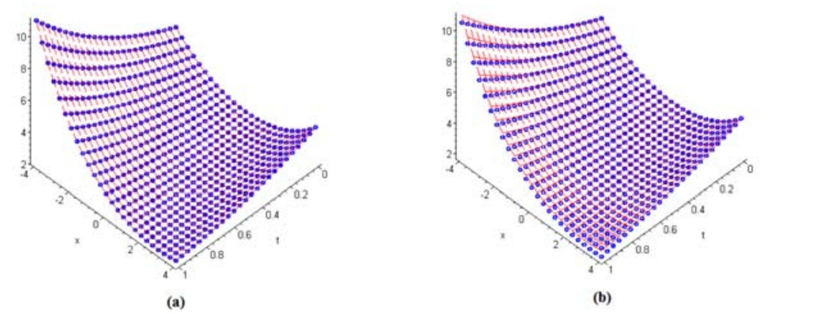

Figure 2. (a) Comparison of the solution of the equation K(3,3,1) by RDTM of order seven (solid line) with the exact solution (point) for c=3.
(b) Comparison of the solution of the equation K(3,3,1) by RDTM of order seven (solid line) with the VIM solution with three terms (point) for c=3.

4. Conclusion

The main goal of this study is to construct an approximate analytical solution for the nonlinear dispersive $K(m,n,1)$equations. We have achieved this goal by applying the reduced differential transform method. Two special cases $K(2,2,1)$ and $K(3,3,1)$ are chosen to illustrate the effectiveness and efficiency of the method. Results are compared with analytical solutions, and some approximation methods such as Adomian decomposition method and variational iteration method. The main advantage of the RDTM is to provide the user an analytical approximation to the solution, in many cases, an exact solution in a rapidly convergent series with elegantly computed terms. By using differential operators only, RDTM needs small size of computation unlike other numerical methods and introduces a significant improvement in solving nonlinear dispersive equations over existing methods. The solution procedure of the RDTM is simpler than the classical differential transform method (DTM) and requires significantly less computational effort. For the initial value problems, RDTM obtains the solution in an infinite power series which can be easily expressed in a closed form that is the exact solution of the problem. The results show that the RDTM is a powerful computational tool for solving nonlinear dispersive equations. It is also a promising method to solve other types of nonlinear equations.

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