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Oscillation Results of Higher Order Nonlinear Neutral Delay Differential Equations

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Abstract. In this paper, we shall consider higher order nonlinear neutral delay differential equation of the type

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)[x(\sigma(t))]^{\alpha} = 0, \quad t \geq t_0, \ n \in \mathbb{N},$$

where $p, q \in C([t_0, \infty), [0, \infty))$, $\tau, \sigma \in C([t_0, \infty), \mathbb{R})$, $\tau(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\sigma(t) < t$, $\lim_{t\to\infty} \sigma(t) = \infty$, and $\alpha \in (0, \infty)$ is a ratio of odd positive integers. We obtain sufficient conditions for the oscillations of all solutions of this equation.

Key words: Oscillation, differential equation, neutral, delay, nonlinear.
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1. Introduction

We consider the following higher order nonlinear neutral delay differential equation:

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)[x(\sigma(t))]^{\alpha} = 0, \quad t \geq t_0, \ n \in \mathbb{N}, \tag{1.1}$$

where $p, q \in C([t_0, \infty), [0, \infty))$, $\tau, \sigma \in C([t_0, \infty), \mathbb{R})$, $\tau(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\sigma(t) < t$, $\lim_{t\to\infty} \sigma(t) = \infty$, and $\alpha \in (0, \infty)$ is a ratio of odd positive integers. If $0 < \alpha < 1$, equation (1.1) is called sublinear equation, when $\alpha > 1$, it is called superlinear equation.
Recently, there have been a lot of studies concerning the behaviour of the oscillatory differential equations, see [1-9] and the reference cited therein. In [2,4,6,8] several authors have investigated the following first order nonlinear delay differential equation of the form,

$$x'(t) + q(t)[x(\sigma(t))]^{\alpha} = 0, \quad t \geq t_0, \tag{1.2}$$

where $q \in C([t_0,\infty),[0,\infty))$, $\sigma \in C([t_0,\infty),\mathbb{R})$, $\sigma(t) < t$, $\lim_{t\to\infty}\sigma(t) = \infty$, and $\alpha \in (0,\infty)$ is a ratio of odd positive integers.
When $0 < \alpha < 1$, it is shown that every solution of sublinear equation (1.2) oscillates if and only if

$$\int_{t=t_0}^{\infty} q(t)dt = \infty. \tag{1.3}$$

When $\alpha = 1$, (1.2) reduces to the linear delay differential equation

$$x'(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0. \tag{1.4}$$

Recently, the oscillatory behavior of (1.4) has been discussed extensively in the literature. A classical result is (see[2,4]) that every solution of (1.4) oscillates if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)ds > \frac{1}{e}.$$

In [6], when $\alpha > 1$, Tang obtained the oscillatory behavior of equation (1.2). It is shown that, let $\sigma(t)$ is continuously differentiable and $\sigma'(t) \geq 0$. Further, suppose that there exist a continuously differentiable function $\varphi(t)$ such that

$$\varphi'(t) > 0 \text{ and } \lim_{t\to\infty}\varphi(t) = \infty,$$

$$\limsup_{t\to\infty}\left[\frac{\alpha\varphi'(\sigma(t))\sigma'(t)}{\varphi'(t)}\right] < 1,$$

and

$$\liminf_{t\to\infty}\left[\frac{q(t)e^{-\varphi(t)}}{\varphi'(t)}\right] > 0.$$

Then every solution of superlinear equation (1.2) oscillates. Furthermore, Tang considered the following special form of (1.2),

$$x'(t) + q(t)[x(t-\sigma)]^{\alpha} = 0, \quad t \geq t_0 \tag{1.5}$$

which was obtained, if exists $\lambda > \sigma^{-1}\ln\alpha$ such that

$$\liminf_{t\to\infty}[q(t)\exp(e^{-\lambda t})] > 0, \tag{1.6}$$

then every solution of (1.5) oscillates.
In [1] Agarwal and Grace, in [3] Grace and Lalli studied oscillatory behavior of certain higher order differential equations.
Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of (1.1).
We need the following result for our subsequent discussion.

Lemma 1.1. (See[8].)Assume that for large t

$$p(s) \neq 0 \text{ for } s \in [t, t^*],$$

where t^* satisfies $\sigma(t^*) = t$. Then

$$x'(t) + p(t)[x(\sigma(t))]^\alpha = 0, \quad t \geq t_0$$

has an eventually positive solution if and only if the corresponding inequality

$$x'(t) + p(t)[x(\sigma(t))]^\alpha \leq 0, \quad t \geq t_0$$

has an eventually positive solution.

Lemma 1.2. (See[5].)Let z be a positive and n-times diferentiable function on $[t_0, \infty)$. If z^n is of constant sign for $n \geq a$ and not identically zero on any interval $[t_*, \infty)$ for some $t_* \geq t_0$, then, there exists a $t_z \geq t_0$ and an integer m, $0 \leq m \leq n$ with $(n+m)$ odd for $z^{(n)}(t) \leq 0$, or $(n+m)$ even for $z^{(n)}(t) \geq 0$, and such that for every $t_z \geq t_0$,

$$m \leq n-1 \text{ implies } (-1)^{m+k} z^{(k)}(t) > 0, \quad k = m, m+1, \cdots, n-1,$$

and

$$m > 0 \text{ implies } z^{(k)}(t) > 0, \qquad k = 0, 1, \cdots, m-1.$$

Lemma 1.3. (See[7].)Let z be as in Lemma 1.2. In addition $\lim_{t\to\infty} z(t) \neq 0$ and $z^{(n-1)}(t) z^{(n)}(t) \leq 0$ for every $t \geq t_z$, then for every λ, $0 < \lambda < 1$, the following hold:

$$z(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t); \text{ for all large } t.$$

2. Sufficient Conditions for Oscillations of Equation (1.1)

Theorem 2.1. (**a**) Let n be even and $0 \leq p(t) < 1$ for $t \geq t_0$. If the diferential equation

$$w'(t) + c(t)[w(\sigma(t))]^\alpha = 0, \tag{2.1}$$

where

$$c(t) = q(t) \left(\frac{\lambda}{(n-1)!} \right)^\alpha [1 - p(\sigma(t))]^\alpha (\sigma(t))^{\alpha(n-1)}. \tag{2.2}$$

is oscillatory, then all solutions of (1.1) are oscillatory.

(**b**) Let n be odd and $0 \leq p(t) \leq P_1 < 1$, where P_1 is a constant. If the diferential equation

$$w'(t) + c(t)[w(\sigma(t))]^{\alpha} = 0, \tag{2.3}$$

where

$$c(t) = q(t)c_2 \left(\frac{\lambda}{(n-1)!}\right)^{\alpha} (\sigma(t))^{\alpha(n-1)}[w(\sigma(t))]^{\alpha} \tag{2.4}$$

is oscillatory, then every solution of (1.1) either oscillates or tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), with $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$, for all $t \geq t_0 \geq T_0$. Setting $z(t) = x(t) + p(t)x(\tau(t))$, we get $z(t) \geq x(t) > 0$ and

$$z^{(n)}(t) = -q(t)x^{\alpha}(\sigma(t)) < 0, \tag{2.5}$$

for $t \geq t_0$. Then by Lemma 1.2, $z^{(k)}(t)$ is of constant sign for $k = 1, 2, 3, ..., n$, and that for $n \geq 2$

$$z^{(n-1)}(t) > 0\,, \quad t \geq t_0 \tag{2.6}$$

We claim that $z'(t) \leq 0$ eventually. This is obvious from equation (1.1) in the case $n = 1$. For $n \geq 2$, we suppose on the contrary, that $z'(t) > 0$ for $t \geq t_1 \geq t_0$. Then

$$(1 - p(t))z(t) \leq z(t) - p(t)z(\tau(t)) = x(t) - p(t)p(\tau(t))x(\tau(\tau(t))) \leq x(t) \tag{2.7}$$

for $t \geq t_2 \geq t_1$. Since $z(t)$ is positive and increasing, it follows from Lemma 1.3 and (2.7), that

$$x(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} x^{(n-1)}(t),\ t \geq t_2, \tag{2.8}$$

Using (2.8), we find for $t \geq t_2 \geq t_1$,

$$q(t)x(\sigma(t)) \geq q(t)\frac{[1 - p(\sigma(t))]\lambda}{(n-1)!}(\sigma(t))^{n-1} z^{(n-1)}(\sigma(t)),$$

and so

$$z^{(n)}(t) \leq -q(t)\left(\frac{[1 - p(\sigma(t))]\lambda}{(n-1)!}\right)^{\alpha} (\sigma(t))^{\alpha(n-1)} \left[z^{(n-1)}(\sigma(t))\right]^{\alpha}.$$

Using the above inequality in (2.5), we see that $z^{(n-1)}(t)$ is an eventually positive (see (2.6)) solution of

$$z^{(n)}(t) + q(t)\left(\frac{[1 - p(\sigma(t))]\lambda}{(n-1)!}\right)^{\alpha} (\sigma(t))^{\alpha(n-1)} \left[z^{(n-1)}(\sigma(t))\right]^{\alpha} \leq 0$$

If we chose $z^{(n-1)}(t) = w(t)$, then

$$w'(t) + q(t)\left(\frac{\lambda}{(n-1)!}\right)^{\alpha}[1 - p(\sigma(t))]^{\alpha}(\sigma(t))^{\alpha(n-1)}w^{\alpha}(\sigma(t)) \leq 0,$$

for some $t_3 \geq t_2$ and hence by (2.6), we have

$$w'(t) + c(t)[w(\sigma(t))]^{\alpha} \leq 0, \text{ for } t \geq t_3.$$

Therefore by Lemma 1.1, (2.1) has eventually positive solution, this is a contradiction. Hence, $z'(t) \leq 0$ eventually.
Since $z'(t) \leq 0$ eventually in Lemma 1.2, we must have $m = 0$ and

$$(-1)^k z^{(k)}(t) > 0, \quad 0 \leq k \leq n-1, \quad t \geq t_0. \tag{2.9}$$

If n is even, (2.9) yields to contradiction (2.6). This proves part (a) of the theorem.
Now, let n be odd. Assume further that $x(t)$ does not tend to zero as $t \to \infty$. As $z'(t) \leq 0$ eventually, we have $z(t) \downarrow c$ as $t \to \infty$, where $0 < c < \infty$. Then, there exists $\varepsilon > 0$ and an integer $t_3 > t_0$ such that

$$0 < \varepsilon < c\frac{1 - P_1}{1 + P_1} < c,$$

and

$$c - \varepsilon < z(t) \leq z(\tau(t)) < c + \varepsilon, \quad t \geq t_3 \tag{2.10}$$

Thus, from (2.7) and (2.10), we find for $t \geq t_3$,

$$x(t) \geq z(t) - p(t)z(\tau(t)) \geq z(t) - P_1 z(\tau(t)) > (c-\varepsilon) - P_1(c+\varepsilon) > c_1 z(t), \tag{2.11}$$

where $c_1 = [(c - \varepsilon) - P_1(c + \varepsilon)]/(c + \varepsilon) \in (0, 1)$. Using (2.11) and Lemma 1.3, we get for $t \geq t_4 \geq t_3$,

$$x(t) > c_1 z(t) \quad > c_1 \tfrac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t) \tag{2.12}$$

By (2.12), we obtain for $t \geq t_5 \geq t_4$,

$$q(t)x(\sigma(t)) \geq q(t)c_1 \frac{\lambda}{(n-1)!}(\sigma(t))^{n-1} z^{(n-1)}(\sigma(t)).$$

There, we have

$$z^{(n)}(t) + q(t)(c_1)^{\alpha}\left(\frac{\lambda}{(n-1)!}\right)^{\alpha}(\sigma(t))^{\alpha(n-1)}\left[z^{(n-1)}(\sigma(t))\right]^{\alpha} \leq 0.$$

Using the above inequality in (2.5), we see that $z^{(n-1)}(t)$ is an eventually positive (see (2.6)) solution of

$$w'(t) + q(t)c_2\left(\frac{\lambda}{(n-1)!}\right)^{\alpha}(\sigma(t))^{\alpha(n-1)}w^{\alpha}(\sigma(t)) \leq 0,$$

where $w(t) = z^{(n-1)}(t)$ and $c_2 = (c_1)^\alpha$. Therefore by Lemma 1.1, (2.1) has eventually positive solution, this is a contradiction. The proof of part (b) is complete.

Theorem 2.2. Let $-1 < -P_2 \le p(t) \le 0$, where $P_2 > 0$ is a constant. If the differential equation

$$w'(t) + c(t)[w(\sigma(t))]^\alpha = 0, \tag{2.13}$$

where

$$c(t) = q(t)\left(\frac{\lambda}{(n-1)!}\right)^\alpha (\sigma(t))^{\alpha(n-1)}[w(\sigma(t))]^\alpha \tag{2.14}$$

is oscillatory, then each monotone solution of (1.1) tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be a monotone solution of (1.1). The case $n = 1$ can be proved easily. Assume that $n \ge 2$ and $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$, for all $t \ge t_0 \ge T_0$. Furter, we assume that $x(t)$ does not tend to zero as $t \to \infty$. Setting $z(t) = x(t) + p(t)x(\tau(t))$, we get $z(t) \le x(t)$, and also inequality (2.5). Since x is monotone, we have either $x'(t) \le 0$ or $x'(t) > 0$ eventually.
We claim that $x'(t) \le 0$ eventually. This is obvious from equation (1.1) in the case $n = 1$. For $n \ge 2$, we suppose on the contrary, that $x'(t) > 0$ for $t \ge t_1 \ge t_0$. Since $-1 < -P_2 \le p(t) \le 0$, we get for $t \ge t_2 \ge t_1$,

$$z(t) \ge x(t) + p(t)x(t) \ge (1 - P_2)x(t) > 0. \tag{2.15}$$

Thus, $z'(t)$ is of one sign, i.e., either $z'(t) \le 0$ or $z'(t) > 0$ holds for $t \ge t_3 \ge t_2$ by Lemma 1.2.
(i) Assume that $z'(t) \le 0$. Then $z(t)$ converges to a constant $z_1 \ge 0$. If $z_1 = 0$, by (2.15), $x(t)$ converges to 0, this contradicts to $x'(t) \ge 0$, $x(t) > 0$. Hence, $z_1 > 0$. Given $\varepsilon_1 \in (0, z_1)$, there exists $t_4 \ge t_3$ such that

$$z_1 - \varepsilon_1 < z\,(t) < z_1 + \varepsilon_1, t \ge t_4. \tag{2.16}$$

Let m be as in Lemma 1.3. For $t \ge t_5 \ge 2^{n-1}t_4$, using (2.16) and Lemma 1.3 successively, we obtain

$$x(t) \ge z(t) \ge \frac{\lambda}{(n-1)!} t^{n-1} z^{(n-1)}(t),\ t \ge t_5. \tag{2.17}$$

By (2.17), we obtain for $t \ge t_6 \ge t_5$,

$$z^{(n)}(t) \le -q(t)\left(\frac{\lambda}{(n-1)!}\right)^\alpha (\sigma(t))^{\alpha(n-1)} \left[z^{(n-1)}(\sigma(t))\right]^\alpha.$$

If we chose $z^{(n-1)}(t) = w(t)$, then

$$w'(t) + q(t)\left(\frac{\lambda}{(n-1)!}\right)^\alpha (\sigma(t))^{\alpha(n-1)}[w(\sigma(t))]^\alpha \le 0.$$

Using the above inequality in (2.5), we see that $z^{(n-1)}(t)$ is an eventually positive (see (2.6)) solution of

$$w'(t) + q(t)\left(\frac{\lambda}{(n-1)!}\right)^{\alpha}(\sigma(t))^{\alpha(n-1)}[w(\sigma(t))]^{\alpha} \leq 0. \tag{2.18}$$

Therefore by Lemma 1.1, (2.13) has eventually positive solution, this is a contradiction.
(ii) Assume that $z'(t) > 0$, $t \geq t_7$. Then by Lemma 1.3, we have for $t \geq t_7$

$$z(t) \geq \frac{\lambda}{(n-1)!}t^{n-1}z^{(n-1)}(t),\ t \geq t_7,$$

and the inequality

$$z^{(n)}(t) + q(t)\left(\frac{\lambda}{(n-1)!}\right)^{\alpha}(\sigma(t))^{\alpha(n-1)}\left[z^{(n-1)}(\sigma(t))\right]^{\alpha} \leq 0,$$

where $z^{(n-1)}(t) = w(t)$, has an eventually positive solution This is a contradiction.
Consequently, $x'(t) \leq 0$ eventually, which tells us that $x(t)$ is nonincreasing and bounded from below, and so coverges to a constant $x_0 \geq 0$. If $x_0 = 0$, then the result is true. Assume that $x_0 > 0$. Then we have

$$\liminf_{t\to\infty} z(t) = (1 + \liminf_{t\to\infty} p(t))x_0 \geq (1 - P_2)x_0 > 0. \tag{2.19}$$

Hence, $z(t)$ is eventualy positive and (2.6) holds. By Lemma 1.2, either $z'(t) < 0$ or $z'(t) > 0$ holds for $t \geq t_8$. Similar to the above proof of (i) and (ii) , we can also obtain contradiction. The case when $x(t)$ is monotone and eventualy negative can be verified similarly. The proof is complete.

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