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Fixed Points of Quasi–Nonexpansive Mappings and Best Approximation

T. D. Narang[1], Sumit Chandok[2]

[1]Department of Mathematics, Guru Nanak Dev University, Amritsar-143005, India
e-mail: tdnarang1948@yahoo.co.in
[2]School of Mathematics and Computer Applications,Thapar University, Patiala-147004, India
e-mail: chansok.s@gmail.com, sumit.chandok@thapar.edu

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Abstract. Using fixed point theory, B. Brosowski [Mathematica (Cluj) 11(1969), 195-220] proved that if T is a nonexpansive linear operator on a normed linear space X, C a T-invariant subset of X and x a T-invariant point, then the set $P_C(x)$ of best C-approximant to x contains a T-invariant point if $P_C(x)$ is non-empty, compact and convex. Subsequently, many generalizations of the Brosowski's result have appeared. In this paper, we also prove some extensions of the results of Brosowski and others for quasi-nonexpansive mappings when the underlying spaces are metric linear spaces or convex metric spaces.

Key words: Best approximation, approximatively compact set, locally convex metric linear space, convex metric space, convex set, starshaped set, nonexpansive map and quasi-nonexpansive map.
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1.Introduction and Preliminaries

Using fixed point theory, Meinardus [8] and Brosowski [2] esdtablished some interesting results on invariant approximation for nonexpansive mappings in normed linear spaces. Various generalizations of their results were later obtained by other authors (see e.g. [6] and [9]). The present paper is also a step in the same direction. We also prove some extensions of their results for quasi-nonexpansive mappings when the underlying spaces are metric linear spaces or convex metric spaces. Our results contain as a special case some of the results proved in [1], [5], [9] and [10].
To start with, we give some basic definitions:
Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be **nonexpansive** on X if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. A point $x \in X$ is said to be a

fixed point of the mapping T if $Tx = x$. Suppose $F(T)$ denotes the set of fixed points of T in X. A mapping $T : X \to X$ is said to be **quasi-nonexpansive** on X if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in X$ and $p \in F(T)$.
A nonexpansive mapping T on X with $F(T) \neq \emptyset$ is quasi-nonexpansive, but not conversely. A linear quasi-nonexpansive mapping on a Banach space is nonexpansive. But there exist (see e.g. [11], p.27) continuous and discontinuous nonlinear quasi-nonexpansive mappings that are not nonexpansive.
For a non-empty subset C of X and $x \in X$, an element $y \in C$ is said to be a **best approximation** to x or a **best C-approximant** to x if

$$d(x, y) = d(x, C) \equiv \inf\{d(x, z) : z \in C\}.$$

The set of all such $y \in C$ is denoted by $P_C(x)$. The set-valued mapping $P_C : X \to 2^C \equiv$ collection of all subsets of C, is called **metric projection**. A sequence $< y_n >$ in C is called a **minimizing sequence** for x if $\lim_{n\to\infty} d(x, y_n) = d(x, C)$. The set C is said to be **approximatively compact** if for each $x \in X$, every minimizing sequence $< y_n >$ in C has a subsequence $< y_{n_i} >$ converging to an element of C.
A subset C of a linear space L is said to be **convex** if $\lambda x + (1 - \lambda)y \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.
The following proposition will be used in the sequel:

Proposition 1. Let C be a non-empty approximatively compact subset of a metric space (X, d), $x \in X$ and P_C be the metric projection of X onto C defined by $P_C(x) = \{y \in C : d(x, y) = d(x, C)\}$. Then $P_C(x)$ is a non-empty compact subset of C.

Proof. By the definition of $d(x, C)$, there is a sequence $< y_n >$ in C such that

$$\lim d(x, y_n) = d(x, C) \tag{1}$$

i.e. $< y_n >$ is a minimizing sequence for x in C. Since C is approximatively compact, there is a subsequence $< y_{n_i} >$ such that $< y_{n_i} > \to y \in C$. Consider

$$\begin{aligned} d(x, y) &= d(x, \lim y_{n_i}) \\ &= \lim d(x, y_{n_i}) \\ &= d(x, C), \text{ by } (1) \end{aligned}$$

i.e. $y \in P_C(x)$ and so $P_C(x)$ is non-empty.
Now we show that $P_C(x)$ is compact. Let $< y_n >$ be a sequence in $P_C(x)$ i.e. $d(x, y_n) = d(x, C)$ for all n and so $\lim d(x, y_n) = d(x, C)$ i.e. (1) is satisfied and so proceeding as above, we get a subsequence $< y_{n_i} >$ of $< y_n >$ converging to an element $y \in P_C(x)$. This shows that $P_C(x)$ is compact.

Note. It can be easily seen (see Singer [13], p.380) that $P_C(x)$ is always a bounded set and is closed if C is closed.
Brosowski [2] proved the following result on invariant approximation:

Theorem 1. Let T be a non-expansive linear operator on a normed linear space X, C a T-invariant subset of X and x a point of $F(T)$. If $P_C(x)$ is non-empty, compact and convex, then $P_C(x) \cap F(T) \neq \emptyset$.
Since a non-expansive mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive and continuous, we have the following extension of Theorem 1 in metric linear spaces:

Theorem 2. Let T be a continuous quasi-nonexpansive mapping on a locally convex metric linear space (X, d). Let C be a T-invariant subset of X and x a point of $F(T)$. If $P_C(x)$ is non-empty, compact and convex, then $P_C(x) \cap F(T) \neq \emptyset$.

Proof. Let $y \in P_C(x)$. Since $d(x, Ty) = d(Tx, Ty) \leq d(x, y) = d(x, C))$, $Ty \in P_C(x)$ as C is T-invariant. Thus $T : P_C(x) \to P_C(x)$. Since $P_C(x)$ is a compact convex subset of a locally convex metric linear space, by Schauder-Tychnoff theorem (see Theorem 2.3 [7]), T has a fixed point in $P_C(x)$ i.e. $P_C(x) \cap F(T) \neq \emptyset$.
Combining Theorem 2 and Proposition 1, we have:

Corollary 1. Let T be a continuous quasi-nonexpansive mapping on a locally convex metric linear space (X, d) and C an approximatively compact T-invariant subset of X. Let x be a point of $F(T)$ and $P_C(x)$ a convex set. Then $P_C(x) \cap F(T) \neq \emptyset$.
Since every normed linear space is a locally convex metric linear space, we have:

Corollary 2 (Corollary 2.5 [5]). Let X be a normed linear space and C an approximatively compact subset of X. If f is a nonexpansive mapping which has a fixed point x in X and the set $P_C(x)$ is convex, then f has a fixed point in C which is also an element of best approximation of x from C.
Since a quasi-nonexpansive mapping is continuous and for a continuous mapping T, $T(P_C(x))$ is compact if $P_C(x)$ is compact, we have another extension of Theorem 1.

Theorem 3. Let T be a quasi-nonexpansive mapping on a locally convex metric linear space (X, d). Let C be a T-invariant subset of X and x a point of $F(T)$. If $P_C(x)$ is a non-empty, closed convex set in X and T is such that $T(P_C(x))$ is contained in a compact set, then $P_C(x) \cap F(T) \neq \emptyset$.

Proof. Since T is quasi-nonexpansive, proceeding as in Theorem 2 we obtain, $T : P_C(x) \to P_C(x)$. Since $P_C(x)$ is a closed convex set and $T(P_C(x))$ is contained in a compact set, T has a fixed point in $P_C(x)$ (Theorem 2.1 (b) [3]) i.e. $P_C(x) \cap F(T) \neq \emptyset$.

Remarks. A metric linear space (X, d) is said to be **convex** if $d(\lambda x+(1-\lambda)y, z)$ for every $x, y, z \in X$ and $0 \leq \lambda \leq 1$. Since for convex metric linear spaces $P_C(x) \subset \partial C \cap C$ (see [12]), for such spaces one can assume in Theorems 2 and 3

that $T : \partial C \to C$ instead of C is T-invariant as the only use made of $T : C \to C$ is to prove that $T : P_C(x) \to P_C(x)$.
Before proving some more extensions of Theorem 1, we recall a few definitions. For a metric space (X, d), a mapping $W : X \times X \times [0, 1] \to X$ is said to be a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure is called a **convex metric space** [14].
A convex metric space (X, d) is said to satisfy **Property (I)** [4] if for all $x, y \in X$ and $\lambda \in [0, 1]$, $d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y)$, where p is arbitrary but fixed point of X.
A subset C of a convex metric space (X, d) is said to be a **convex set** [14] if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. The set C is said to be **starshaped** [4] if there exists $p \in C$ such that $W(x, p, \lambda) \in C$ for all $x \in C$ and $\lambda \in [0, 1]$.
A normed linear space and each of its convex subsets are simple examples of convex metric spaces which are not normed linear spaces (see [4]). Property (I) is always satisfied in a normed linear space.
We have the following extension of Theorem 1 in convex metric spaces:

Theorem 4. Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) satisfying Property (I), C a T-invariant subset of X and x a point of $F(T)$. If $P_C(x)$ is non-empty, compact and starshaped, and T is nonexpansive on $P_C(x)$, then $P_C(x) \cap F(T) \neq \emptyset$.

Proof. Since T is quasi-nonexpansive, as proved in Theorem 2, $T : P_C(x) \to P_C(x)$. Since $P_C(x)$ is non-empty compact and starshaped, and $T : P_C(x) \to P_C(x)$ is nonexpansive, T has a fixed point in $P_C(x)$ (Theorem 3.4 [4]) and so $P_C(x) \cap F(T) \neq \emptyset$.
Since every normed linear space is a convex metric space with Property (I), we have:

Corollary 3 (Theorem [10]). Let T be a nonexpansive operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is non-empty, compact and starshaped, then $P_C(x) \cap F(T) \neq \emptyset$.
Using Proposition 1, we have:

Theorem 5. Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) satisfying Property (I) and C a T-invariant approximatively compact subset of X. Let x be a point of $F(T)$ and $P_C(x)$ a starshaped set. If T is nonexpansive on $P_C(x)$, then $P_C(x) \cap F(T) \neq \emptyset$.
Since every normed linear space is a convex metric space satisfying Property (I), we have:

Corollary 4 (Theorem 5 [9]). Let T be a quasi-nonexpansive operator on a normed linear space X and C an approximatively compact T-invariant subset of X. Let x be a point of $F(T)$ and $P_C(x)$ a starshaped set. If T is nonexpansive on $P_C(x)$, then $P_C(x) \cap F(T) \neq \emptyset$.
To obtain another extension of Theorem 1, we need the following:

Lemma 1. Let (X, d) be a metric space and $T : X \to X$ a quasi-nonexpansive mapping with a fixed point $u \in X$. If C is a closed T-invariant subset of X and the restriction T/C is a compact mapping, then the set $P_C(u)$ of best approximations is non-empty.
This result was proved in [6]-Theorem 3 for nonexpansive mapping $T : X \to X$ and it can be seen that the proof is valid when the mapping is quasi-nonexpansive.

Lemma 2(Theorem 3 [1]). Let X be a convex metric space satisfying Property (I) and E a closed and starshaped subset of X. If T is a nonexpansive self mapping on E and closure of $T(E)$ is compact then T has a fixed point in E.
Using Lemmas 1 and 2, we have the following generalization of Theorem 1 for convex metric spaces:

Theorem 6. Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) satisfying Property (I). Let C be a closed T-invariant subset of X with T/C compact and x a T-invariant point. If T is nonexpansive on $P_C(x)$ and $P_C(x)$ is a starshaped set, then $P_C(x) \cap F(T) \neq \emptyset$.

Proof. By Lemma 1, $P_C(x)$ is non-empty. We show that $P_C(x)$ is T-invariant. Let $r = d(x, C)$ and $y \in P_C(x)$. Then

$$\begin{aligned} r &\leq d(x, Ty) \text{ as } y \in C \Rightarrow Ty \in C \\ &\leq d(x, y) \text{ as } T \text{ is quasi-nonexpansive} \\ &= r. \end{aligned}$$

Therefore $d(x, Ty) = r$ and so $Ty \in P_C(x)$. This proves that $T : P_C(x) \to P_C(x)$.
If $P_C(x)$ is a singleton, then $P_C(x) = \{y\}$ and so $Ty = y$ i.e. the result is proved in this case. So, suppose $P_C(x)$ contains more than one point. Since C is closed, $P_C(x)$ is closed. Also $P_C(x)$ is always bounded. Since T/C is compact, $\overline{T(P_C(x))}$ is compact. Since $P_C(x)$ is starshaped and $T : P_C(x) \to P_C(x)$ is nonexpansive, T has a fixed point in $P_C(x)$ by Lemma 2 and so $P_C(x) \cap F(T) \neq \emptyset$.
Since every convex set is starshaped, we get:

Corollary 5 (Theorem 10 [1]). Let (X, d) be a convex metric space satisfying Property (I) and T a nonexpansive mapping on X. Let C be a closed T-invariant subset of X with T/C compact and x a T-invariant point. If $P_C(x)$ is non-empty, convex and compact, then it contains a T-invariant point.

Remarks. Since in a convex metric space, $P_C(x) \subseteq \partial C \cap C$, the condition '$C$ is T-invariant' in Theorems 3 to 5 can be weakened to $T : \partial C \to C$ as the only use made of $T : C \to C$ is to prove that $T : P_C(x) \to P_C(x)$.

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