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An Application of Exp–Function Method to the Generalized Burger's–Huxley Equation

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Abstract. In this work, we implement a relatively new analytical technique, the Exp–Function method, for solving nonlinear equations and absolutely special form of Generalized Burger's–Huxley equation which may contain high nonlinear terms. This method can be used as an alternative to obtain analytical and approximate solutions of different types of fractional differential equations which applied in engineering mathematics. For more illustration of the efficiency and reliability of Exp method some numerical examples are presented. It is predicted that Exp–Function method can be found widely applicable in engineering.

Key words: Exp–Function method; Generalized Burger's–Huxley equation; Nonlinear Partial Differential equations.

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1. Introduction

Many problems in natural and engineering sciences are modeled by partial differential equations. Nonlinear phenomena play important roles in applied mathematics, physics and also in engineering problems in which each parameter varies depending on different factors. Solving nonlinear equations may guide authors to know the described process deeply and sometimes leads them to know some facts which are not simply understood through common observations. Moreover, obtaining exact solutions for these problems is a great purpose which has

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been quite untouched. However, in recent years, numerical analysis [1] has considerably been developed to be used for nonlinear partial equations such as Generalized Burger's–Huxley equation that shows a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transports [2]. The solitary waves and the existed property of a finite number of quantities (with physical interpretation) are conserved by the solutions.
The Generalized Burger's–Huxley equation is in the form of:

$$\begin{aligned} & u_t + \alpha u^{\delta} u_x - u_{xx} = \beta u(1 - u^{\delta})(u^{\delta} - \gamma), \\ & \forall \;\; 0 \leq x \leq 1, \quad t \geq 0, \quad u(x, 0) = f(x) \end{aligned} \tag{1}$$

When $\alpha = 0$, $\delta = 1$ Eq. (1) is reduced to the Huxley equation which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals [3] and the subscripts t and x denote differentiation with respect to time and space, respectively.
In addition, in recent years, scientists have presented some new methods for solving nonlinear partial differential equations; for instance, Bäcklund transformation method [4], Lie group method [5], Adomian's decomposition method [6], inverse scattering method [7], Hirota's bilinear method [8], homotopy analysis method [9] and He's Homotopy perturbation method[10-11], He's Variational iteration method[12-13] and Exp-Function method [14-15].
In this letter, we purpose to present implementation of Exp–Function method to generalized nonlinear Generalized Burger's–Huxley equation. Having the available exact solution of special form of corresponding equations [16] would provide us to benefit an admissible comparison of the results which supports the applicability, accuracy and efficiency of the proposed methods.

2. Basic Idea of Exp-Function Method

We first consider nonlinear equation form:

$$N(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, \ldots) = 0. \tag{2}$$

Introduction a complete variation defines as:

$$\eta = \mu(x + \omega t), u = U(\eta), \tag{3}$$

and therefore, the Eq(1) is the construction of ODE of form:

$$N(U, \omega\mu U', \mu U'^2 U''^2 \mu^2 U'', \omega\mu^2 U'', \ldots) = 0. \tag{4}$$

and then solution of $U(\eta)$ is form:

$$U(\eta)=\frac{\sum_{n=-c}^{d}a_n\exp(n\eta)}{\sum_{m=-p}^{q}b_m\exp(m\eta)}=\frac{a_c\exp(c\eta)+...+a_{-d}\exp(-d\eta)}{b_p\exp(p\eta)+...+b_{-q}\exp(-q\eta)} \tag{5}$$

where c, d, p and q are positive integers which are unknown to be further determined, a_n and b_m are unknown constants.

3. Application of Exp–Function Method to Generalized Burger's–Huxley Equation

We begin with the transformation $v(x,t)=u^{\delta}(x,t)$ or $u(x,t)=v^{1/\delta}(x,t)$, and have:

$$\begin{aligned}&u_t=\frac{1}{\delta}v^{\frac{1}{\delta}-1}v_t,\quad \alpha u^{\delta}u_x=\alpha v\frac{1}{\delta}v^{\frac{1}{\delta}-1}v_x,\\&u_{xx}=\frac{1}{\delta}[v^{\frac{1}{\delta}-1}v_{xx}+(\frac{1}{\delta}-1)v^{\frac{1}{\delta}-2}v_x^2],\\&\beta u(1-u^{\delta})(u^{\delta}-\gamma)=\beta v^{\frac{1}{\delta}}(1-v)(v-\gamma).\end{aligned} \tag{6}$$

Substituting these results into Eq. (1), we have:

$$\frac{1}{\delta}v^{\frac{1}{\delta}-1}[v_t+\alpha v v_x-v_{xx}-(\frac{1}{\delta}-1)v_x^2v^{-1}]=\beta v^{\frac{1}{\delta}}(1-v)(v-\gamma) \tag{7}$$

Eq. (7) is simplified as follows:

$$v_t+\alpha v v_x-v_{xx}-(\frac{1}{\delta}-1)v_x^2v^{-1}=\beta\delta v(1-v)(v-\gamma) \tag{8}$$

Introducing a complex variation η defined as Eq.(3), Eq. (8) becomes an ordinary differential equation which reforms to:

$$\omega\mu V'+\alpha\mu V V'^2(V''+(\frac{1}{\delta}-1)V'^2V^{-1})-\beta\delta V(1-V)(V-\gamma)=0 \tag{9}$$

In order to determine values of c and p, we balance the linear term of the highest order V'' with the highest order nonlinear term V^3 in Eq. (9), we have:

$$V''=\frac{c_1\exp\left[(c+3p)\eta\right]+...}{c_2\exp\left[4p\eta\right]+...}, \tag{10}$$

$$V^3=\frac{c_3\exp\left[(3c+p)\eta\right]+...}{c_4\exp\left[4p\eta\right]+...} \tag{11}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp–function in Eq. (10) and (11), we have:

$$c + 3p = 3c + p \tag{12}$$

which leads to the result:

$$p = c \tag{13}$$

Similarly to determine values of d and q, we balance the linear term of lowest order in Eq. (9)

$$V'' = \frac{... + d_1 \exp\left[-\left(d + 3q\right)\eta\right]}{... + d_2 \exp\left[-4q\eta\right]} \tag{14}$$

and

$$V^3 = \frac{... + d_3 \exp\left[-\left(3d + q\right)\eta\right]}{... + d_4 \exp[-4q\eta]} \tag{15}$$

where d_i are determined coefficients only for simplicity. Balancing lowest order of Exp–Function into Eqs. (14) and (15), we have:

$$-\left(d + 3q\right) = -\left(3d + q\right) \tag{16}$$

This leads to the result:

$$q = d \tag{17}$$

Case 1: $p = c = 1$**,** $d = q = 1$
According to this case, Eq. (5) reduces to:

$$V(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)} \tag{18}$$

Substituting Eq. (18) into Eq. (9), and by the help of MAPLE, we have:

$$\begin{aligned} &\tfrac{1}{A}\{C_4 e^{4\eta} + C_3 e^{3\eta} + C_2 e^{2\eta} + C_1 e^{\eta} + C_0 + C_{-1} e^{-\eta} \\ &+ C_{-2} e^{-2\eta} + C_{-3} e^{-3\eta} + C_{-4} e^{-4\eta})\} = 0, \end{aligned} \tag{19}$$

and C_n are coefficients of $\exp(n\eta)$. All mentioned relations and equations which are unknown in this case are added to this paper in appendix.
Equating the coefficients of $\exp(n\eta)$ must be zero, and therefore we have:

$$C_1=0, C_2=0, C_3=0, C_4=0, C_0=0, C_{-1}=0, C_{-2}=0, C_{-3}=0, C_{-4}=0. \tag{20}$$

Solving the system Eq. (20), simultaneously yields the following sets of nontrivial solutions:

$$\begin{aligned} &b_0=0, \quad a_{-1}=0, \quad b_{-1}=b_{-1}, \quad a_0=0, \quad b_1=1 \quad a_1=\gamma, \\ &\mu=-\frac{\delta\gamma[\alpha-\sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)}, \\ &\omega=2\gamma[\frac{-\alpha}{2(1+\delta)}+\frac{(1+\delta-\gamma)\beta}{\alpha\gamma+\gamma\sqrt{\alpha^2+4\beta(1+\delta)}}]= \\ &\frac{-\gamma\alpha}{(1+\delta)}-\frac{(1+\delta-\gamma)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}. \end{aligned} \tag{21}$$

$$\begin{aligned} &b_0=a_0+\frac{b_{-1}}{a_0}, \quad a_{-1}=0, \quad b_{-1}=b_{-1}, \quad a_0=a_0, \quad b_1=1 \quad a_1=1, \\ &\mu=-\frac{\delta[\alpha-\sqrt{\alpha^2+4\beta(1+\delta)}]}{2(1+\delta)}, \\ &\omega=2[\frac{-\alpha}{2(1+\delta)}+\frac{(\gamma+\gamma\delta-1)\beta}{\alpha+\sqrt{\alpha^2+4\beta(1+\delta)}}]= \\ &\frac{-\alpha}{(1+\delta)}-\frac{(\gamma+\gamma\delta-1)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}. \end{aligned} \tag{22}$$

$$\begin{aligned} &b_0=b_0,\ a_{-1}=0,\ b_{-1}=-\frac{a_0(a_0-\gamma b_0)}{\gamma^2},\ a_0=a_0,\ b_1=1 \quad a_1=\gamma, \\ &\mu=-\frac{\delta\gamma[\alpha-\sqrt{\alpha^2+4\beta(1+\delta)}]}{2(1+\delta)}, \\ &\omega=2\gamma[\frac{-\alpha}{2(1+\delta)}+\frac{(1+\delta-\gamma)\beta}{\alpha\gamma+\gamma\sqrt{\alpha^2+4\beta(1+\delta)}}]= \\ &\frac{-\alpha\gamma}{(1+\delta)}-\frac{(1+\delta-\gamma)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}. \end{aligned} \tag{23}$$

$$\begin{aligned} &b_0=b_0, \quad a_{-1}=0, \quad b_{-1}=0, \quad a_0=0, \quad b_1=1 \quad a_1=1, \\ &\mu=-\frac{\delta[\alpha-\sqrt{\alpha^2+4\beta(1+\delta)}]}{2(1+\delta)}, \\ &\omega=2[\frac{-\alpha}{2(1+\delta)}+\frac{(\gamma+\gamma\delta-1)\beta}{\alpha+\sqrt{\alpha^2+4\beta(1+\delta)}}]= \\ &\frac{-\alpha}{(1+\delta)}-\frac{(\gamma+\gamma\delta-1)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}. \end{aligned} \tag{24}$$

$$(25)\quad \begin{aligned}&b_0=0,\quad a_{-1}=0,\quad b_{-1}=b_{-1},\quad a_0=0,\quad b_1=1\quad a_1=1,\\ &\mu=-\frac{\delta[\alpha-\sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)},\\ &\omega=2[\frac{-\alpha}{2(1+\delta)}+\frac{(\gamma+\gamma\delta-1)\beta}{\alpha+\sqrt{\alpha^2+4\beta(1+\delta)}}]=\\ &\frac{-\alpha}{(1+\delta)}-\frac{(\gamma+\gamma\delta-1)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.\end{aligned}$$

$$(26)\quad \begin{aligned}&b_0=b_0,\ a_{-1}=a_0(b_0-a_0),\ b_{-1}=a_0(b_0-a_0),\ a_0=a_0,\ b_1=1\qquad a_1=0,\\ &\mu=-\frac{\delta[\alpha+\sqrt{\alpha^2+4\beta(1+\delta)}]}{2(1+\delta)},\\ &\omega=2[\frac{-\alpha}{2(1+\delta)}+\frac{(\gamma+\gamma\delta-1)\beta}{\alpha-\sqrt{\alpha^2+4\beta(1+\delta)}}]=\\ &\frac{-\alpha}{(1+\delta)}-\frac{(\gamma+\gamma\delta-1)(\alpha+\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.\end{aligned}$$

$$(27)\quad \begin{aligned}&b_0=0,\quad a_{-1}=\gamma b_{-1},\quad b_{-1}=b_{-1},\quad a_0=a_0,\quad b_1=1\quad a_1=0,\\ &\mu=\frac{\delta\gamma[\alpha-\sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)},\\ &\omega=2\gamma[\frac{\alpha}{2(1+\delta)}-\frac{(1+\delta-\gamma)\beta}{\alpha\gamma-\gamma\sqrt{\alpha^2+4\beta(1+\delta)}}]=\\ &\omega=\frac{\alpha\gamma}{(1+\delta)}+\frac{(1+\delta-\gamma)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.\end{aligned}$$

$$(28)\quad \begin{aligned}&b_0=0,\quad a_{-1}=\gamma b_{-1},\quad b_{-1}=b_{-1},\quad a_0=a_0,\quad b_1=1\quad a_1=0,\\ &\mu=\frac{\delta\gamma[\alpha+\sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)},\\ &\omega=2\gamma[\frac{-\alpha}{2(1+\delta)}+\frac{(1+\delta-\gamma)\beta}{\alpha\gamma-\gamma\sqrt{\alpha^2+4\beta(1+\delta)}}]=\\ &\frac{-\alpha\gamma}{(1+\delta)}-\frac{(1+\delta-\gamma)(\alpha+\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.\end{aligned}$$

Substituting these results into Eq. (18), and $\eta=\mu(x+\omega t)$we obtain the following generalized solitonary solutions of Eq. (8) solution

$$(29)\quad \begin{aligned}&v_1(x,t)=\frac{\gamma\exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)]+b_{-1}\exp[-\mu(x+\omega t)]}\\ &\mu=-\frac{\delta\gamma[\alpha-\sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)},\\ &\omega=\frac{-\gamma\alpha}{(1+\delta)}-\frac{(1+\delta-\gamma)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.\end{aligned}$$

where, b_{-1} is arbitrary constant parameter. Substituting $b_{-1} = 1$ into Eq.(29), we have the following closed form of Eq. (8) and Eq. (1)

(30)
$$
\begin{aligned}
&v_1(x,t) = \frac{\gamma \exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + \exp[-\mu(x+\omega t)]} = \\
&\frac{\gamma}{2}[1 + \tanh(\mu(x+\omega t)], \\
&u_1(x,t) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\mu(x+\omega t)]^{\frac{1}{\delta}} \\
&\mu = -\frac{\delta\gamma[\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)}]}{4(1+\delta)}, \\
&\omega = \frac{-\gamma\alpha}{(1+\delta)} - \frac{(1+\delta-\gamma)(\alpha - \sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned}
$$

(31)
$$
\begin{aligned}
&v_2(x,t) = \frac{\exp[\mu(x+\omega t)] + a_0}{\exp[\mu(x+\omega t)] + a_0 + \frac{b_{-1}}{a_0} + b_{-1}\exp[-\mu(x+\omega t)]} \\
&u_2(x,t) = (\frac{\exp[\mu(x+\omega t)] + a_0}{\exp[\mu(x+\omega t)] + a_0 + \frac{b_{-1}}{a_0} + b_{-1}\exp[-\mu(x+\omega t)]})^{\frac{1}{\delta}} \\
&\mu = -\frac{\delta[\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)}]}{2(1+\delta)}, \\
&\omega = \frac{-\alpha}{(1+\delta)} - \frac{(\gamma+\gamma\delta-1)(\alpha - \sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned}
$$

(32)
$$
\begin{aligned}
&v_3(x,t) = \frac{\gamma\exp[\mu(x+\omega t)] + a_0}{\exp[\mu(x+\omega t)] + b_0 - \frac{a_0(a_0-\gamma b_0)}{\gamma^2}\exp[-\mu(x+\omega t)]} \\
&u_3(x,t) = (\frac{\gamma\exp[\mu(x+\omega t)] + a_0}{\exp[\mu(x+\omega t)] + b_0 - \frac{a_0(a_0-\gamma b_0)}{\gamma^2}\exp[-\mu(x+\omega t)]})^{\frac{1}{\delta}} \\
&\mu = -\frac{\delta\gamma[\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)}]}{2(1+\delta)}, \\
&\omega = \frac{-\alpha\gamma}{(1+\delta)} - \frac{(1+\delta-\gamma)(\alpha - \sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned}
$$

(33)
$$
\begin{aligned}
&v_4(x,t) = \frac{\exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_0} \\
&u_4(x,t) = (\frac{\exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_0})^{\frac{1}{\delta}} \\
&\mu = -\frac{\delta[\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)}]}{2(1+\delta)}, \\
&\omega = \frac{-\alpha}{(1+\delta)} - \frac{(\gamma+\gamma\delta-1)(\alpha - \sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned}
$$

(34)
$$
\begin{aligned}
v_5(x,t) &= \frac{\exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_{-1}\exp[-\mu(x+\omega t)]}\\
u_5(x,t) &= (\frac{\exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_{-1}\exp[-\mu(x+\omega t)]})^{\frac{1}{\delta}}\\
\mu &= -\frac{\delta[\alpha - \sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)},\\
\omega &= \frac{-\alpha}{(1+\delta)} - \frac{(\gamma+\gamma\delta-1)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)},
\end{aligned}
$$

where, b_{-1} is arbitrary constant parameter. Substituting $b_{-1} = 1$ into Eq.(34), we have the following closed form of Eq. (8) and Eq. (1)

(35)
$$
\begin{aligned}
v_5(x,t) &= \frac{\exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + \exp[-\mu(x+\omega t)]} =\\
&\frac{1}{2} + \frac{1}{2}\tanh(\mu(x+\omega t))\\
u_5(x,t) &= [\frac{1}{2} + \frac{1}{2}\tanh(\mu(x+\omega t)]^{\frac{1}{\delta}}\\
\mu &= -\frac{\delta[\alpha - \sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)},\\
\omega &= \frac{-\alpha}{(1+\delta)} - \frac{(\gamma+\gamma\delta-1)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned}
$$

(36)
$$
\begin{aligned}
v_6(x,t) &= \frac{a_0 + a_0(b_0-a_0)\exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_0 + a_0(b_0-a_0)\exp[-\mu(x+\omega t)]}\\
u_6(x,t) &= (\frac{a_0 + a_0(b_0-a_0)\exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_0 + a_0(b_0-a_0)\exp[-\mu(x+\omega t)]})^{\frac{1}{\delta}}\\
\mu &= -\frac{\delta[\alpha + \sqrt{\alpha^2+4\beta(1+\delta)}]}{2(1+\delta)},\\
\omega &= \frac{-\alpha}{(1+\delta)} - \frac{(\gamma+\gamma\delta-1)(\alpha+\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned}
$$

(37)
$$
\begin{aligned}
v_7(x,t) &= \frac{a_0 + \gamma b_{-1}\exp[-\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_{-1}\exp[-\mu(x+\omega t)]}\\
u_7(x,t) &= (\frac{a_0 + \gamma b_{-1}\exp[-\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_{-1}\exp[-\mu(x+\omega t)]})^{\frac{1}{\delta}}\\
\mu &= \frac{\delta\gamma[\alpha - \sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)},\\
\omega &= \frac{\alpha\gamma}{(1+\delta)} + \frac{(1+\delta-\gamma)(\alpha-\sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)},
\end{aligned}
$$

where, b_{-1}, a_0 are arbitrary constant parameters. Substituting $b_{-1} = 1,\ a_0 = 0$ into Eq.(37), we have the following closed form of Eq. (8) and Eq. (1)

$$
\begin{aligned}
&v_7(x,t) = \frac{\gamma \exp[-\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + \exp[-\mu(x+\omega t)]} = \\
&\frac{\gamma}{2}[1+\tanh(-\mu(x+\omega t)], \\
&u_7(x,t) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(-\mu(x+\omega t)]^{\frac{1}{\delta}} \\
&\mu = \frac{\delta\gamma[\alpha - \sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)}, \\
&\omega = \frac{\alpha\gamma}{(1+\delta)} + \frac{(1+\delta-\gamma)(\alpha - \sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
&v_8(x,t) = \frac{a_0 + \gamma b_{-1}\exp[-\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_{-1}\exp[-\mu(x+\omega t)]} \\
&u_8(x,t) = (\frac{a_0 + \gamma b_{-1}\exp[-\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + b_{-1}\exp[-\mu(x+\omega t)]})^{\frac{1}{\delta}} \\
&\mu = \frac{\delta\gamma[\alpha + \sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)}, \\
&\omega = \frac{-\alpha\gamma}{(1+\delta)} - \frac{(1+\delta-\delta)(\alpha + \sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)},
\end{aligned} \tag{39}
$$

where, b_{-1}, a_0 are arbitrary constant parameters. Substituting $b_{-1} = 1$, $a_0 = 0$ into Eq.(39), we have the following closed form of Eq. (8) and Eq. (1)

$$
\begin{aligned}
&v_8(x,t) = \frac{\gamma \exp[-\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + \exp[-\mu(x+\omega t)]} = \\
&\frac{\gamma}{2}[1+\tanh(-\mu(x+\omega t)], \\
&u_8(x,t) = [\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(-\mu(x+\omega t)]^{\frac{1}{\delta}} \\
&\mu = \frac{\delta\gamma[\alpha + \sqrt{\alpha^2+4\beta(1+\delta)}]}{4(1+\delta)}, \\
&\omega = \frac{-\alpha\gamma}{(1+\delta)} - \frac{(1+\delta-\gamma)(\alpha + \sqrt{\alpha^2+4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned} \tag{40}
$$

Case2: $p = c = 2$,$d = q = 2$

As mentioned above the values of c and d can be freely chosen, we set $p = c = 2$ and $d = q = 2$, then the trial function, Eq. (4) becomes:

$$
V(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{b_2\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\eta)} \tag{41}
$$

There are some free parameters in Eq. (41), we set $b_2 = 1$, $a_1 = 0, a_{-1} = 0$ and $b_1 = 0$ for simplicity, the trial function, Eq. (41) is simplified as follows:

$$V(\eta) = \frac{a_2 \exp(2\eta) + a_0 + a_{-2} \exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)} \tag{42}$$

By the same manipulation as illustrated above, we obtain:

$$\begin{aligned}
& b_0 = 0,\ a_{-2} = 0,\ b_{-2} = b_{-2},\ a_0 = 0,\ b_2 = 1,\ a_2 = \gamma,\ \ b_{-1} = 0, \\
& \mu = -\frac{\delta\gamma[\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)}]}{8(1+\delta)}, \\
& \omega = \gamma[\frac{-\alpha}{2(1+\delta)} + \frac{(1+\delta-\gamma)\beta}{\alpha\gamma + \gamma\sqrt{\alpha^2 + 4\beta(1+\delta)}}] = \\
& \frac{-\gamma\alpha}{2(1+\delta)} - \frac{(1+\delta-\gamma)(\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)})}{4(1+\delta)},
\end{aligned} \tag{43}$$

where, $b_{-2}$is arbitrary constant parameter. Substituting Eq. (43) into (42) yields the following solution:

$$\begin{aligned}
& V(\eta) = \frac{\gamma \exp(2\mu\eta)}{\exp(2\mu\eta) + b_{-2} \exp(-2\mu\eta)} \\
& \mu = -\frac{\delta\gamma[\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)}]}{8(1+\delta)}, \\
& \omega = \frac{-\gamma\alpha}{2(1+\delta)} - \frac{(1+\delta-\gamma)(\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)})}{4(1+\delta)}.
\end{aligned} \tag{44}$$

Or

$$\begin{aligned}
& v(x,t) = \frac{\gamma \exp[2\mu(x+\omega t)]}{\exp[2\mu(x+\omega t)] + b_{-2} \exp[-2\mu(x+\omega t)]} \\
& \mu = -\frac{\delta\gamma[\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)}]}{8(1+\delta)}, \\
& \omega = \frac{-\gamma\alpha}{2(1+\delta)} - \frac{(1+\delta-\gamma)(\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)})}{4(1+\delta)}.
\end{aligned} \tag{45}$$

Substituting $b_{-2} = 1$ into Eq.(45), we have the following closed form of Eq. (8) and Eq. (1)

$$\begin{aligned}
& v(x,t) = \frac{\gamma \exp[\mu(x+\omega t)]}{\exp[\mu(x+\omega t)] + \exp[-\mu(x+\omega t)]} = \\
& \frac{\gamma}{2}[1 + \tanh(\mu(x+\omega t)], \\
& u(x,t) = [\frac{\gamma}{2} + \frac{\gamma}{2} \tanh(\mu(x+\omega t)]^{\frac{1}{\delta}} \\
& \mu = -\frac{\delta\gamma[\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)}]}{4(1+\delta)}, \\
& \omega = \frac{-\gamma\alpha}{(1+\delta)} - \frac{(1+\delta-\gamma)(\alpha - \sqrt{\alpha^2 + 4\beta(1+\delta)})}{2(1+\delta)}.
\end{aligned} \tag{46}$$

4. Conclusion

In this Letter, the Exp–Function method was used for finding solutions of Generalized Burger's–Huxley equation which may contain high nonlinear terms. It can be concluded that the Exp–Function method is very powerful and efficient technique in finding exact solutions for wide class of problems.
The Exp–Function method has many merits and more advantages than exact solutions. Calculations in the Exp–Function method are simple and straightforward. The reliability of the method and reduction in the size of computational domain gives this method a wider applicability. The results show that Exp–Function method is a powerful mathematical tool for solving systems of nonlinear partial differential equations having wide applications in engineering.

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Appendix:

Case 1, Generalized Burger's–Huxley equation:

$$A = (e^{\eta} + b_0 + b_{-1}e^{-\eta})^4,$$

$$C_4 = -\beta\delta^2 a_1^3 + \beta\gamma\delta^2 a_1^2 - \beta\gamma\delta^2 a_1^3 + \beta\delta^2 a_1^4 = 0,$$

$$\begin{aligned}C_3 = {} & \mu^2\delta a_1^2 b_0 - \beta\gamma\delta^2 a_1^3 b_0 + 2\beta\gamma\delta^2 a_1^2 b_0 - \alpha\mu\delta a_1^2 a_0 + \omega\mu\delta a_1^2 b_0 - 3\beta\gamma\delta^2 a_1^2 a_0 \\ & -3\beta\delta^2 a_1^2 a_0 + 2\beta\gamma\delta^2 a_0 a_1 + 4\beta\delta^2 a_1^3 a_0 - \omega\mu\delta a_0 a_1 - \beta\delta^2 a_1^3 b_0 - \mu^2\delta a_0 a_1 \\ & +\alpha\mu\delta a_1^3 b_0 = 0,\end{aligned}$$

$$\begin{aligned}C_2 = {} & 6\beta\delta^2 a_1^2 a_0^2 - 3\beta\delta^2 a_1^2 a_{-1} + 4\beta\delta^2 a_1^3 a_{-1} - 4\mu^2\delta a_{-1} a_1 - 3\beta\delta^2 a_1^2 a_0 b_0 \\ & +2\mu^2 a_1 b_0 a_0 + 2\omega\mu\delta a_1^2 b_{-1} + 4\mu^2\delta a_1^2 b_{-1} + \omega\mu\delta a_1^2 b_0^2 + 2\alpha\mu\delta a_1^3 b_{-1} \\ & -2\omega\mu\delta a_{-1} a_1 - 2\alpha\mu\delta a_1 a_0^2 - \omega\mu\delta a_0^2 - 2\alpha\mu\delta a_1^2 a_{-1} + 2\alpha\mu\delta a_1^2 a_0 b_0 \\ & +\beta\gamma\delta^2 a_0^2 - 3\beta\gamma\delta^2 a_1^2 a_{-1} + 4\beta\gamma\delta^2 a_1 b_0 a_0 - \beta\gamma\delta^2 a_1^3 b_{-1} - 3\beta\gamma\delta^2 a_1^2 a_0 b_0 \\ & -\mu^2 a_0^2 + 2\beta\gamma\delta^2 a_1^2 b_{-1} + 2\beta\gamma\delta^2 a_{-1} a_1 + \beta\gamma\delta^2 a_1^2 b_0^2 - 3\beta\gamma\delta^2 a_1 a_0^2 - 3\beta\delta^2 a_1 a_0^2 \\ & -\beta\delta^2 a_1^3 b_{-1}\mu^2 a_1^2 b_0^2 = 0,\end{aligned}$$

$$\begin{aligned}C_1 = {} & -3\beta\delta^2 a_0^2 a_1 b_0 - 3\beta\gamma\delta^2 a_0^2 a_1 b_0 - 6\beta\gamma\delta^2 a_1 a_0 a_{-1} - 6\beta\delta^2 a_1 a_0 a_1 \\ & +12\beta\delta^2 a_1^2 a_0 a_{-1} - 3\beta\delta^2 a_1^2 a_0 b_{-1} - \beta\delta^2 a_0^3 - 4\mu^2 a_{-1} a_0 + 4\beta\delta^2 a_1 a_0^3 + 5\alpha\mu\delta a_1^2 a_0 b_{-1} \\ & -\omega\mu\delta a_0^2 b_0 + \alpha\mu\delta a_1^2 a_{-1} b_0 - 6\alpha\mu\delta a_1 a_0 a_{-1} + 4\mu^2 a_2 b_0 a_{-1} + \omega\mu\delta a_1 b_0^2 a_0 - 3\omega\mu\delta a_{-1} a_0 \\ & -\mu^2\delta a_{-1} a_0 + \mu^2\delta a_1^2 b_0 b_{-1} + 6\mu^2\delta a_1 b_{-1} a_0 - 6\mu^2\delta a_1 b_0 a_{-1} + \alpha\mu\delta a_0^2 a_1 b_0 - \alpha\mu\delta a_0^3 \\ & -2\omega\mu\delta a_{-1} a_1 b_0 - 4\mu^2 a_1^2 b_0 b_{-1} + 4\mu^2 a_1 b_{-1} a_0 + \mu^2\delta a_0^2 b_0 - \mu^2\delta a_1 b_0^2 a_0 + 2\omega\mu\delta a_1 b_{-1} a_0 \\ & +3\omega\mu\delta a_1^2 b_0 b_{-1} + 2\beta\gamma\delta^2 a_1 b_0^2 a_0 + 2\beta\gamma\delta^2 a_1^2 b_0 b_{-1} - \beta\gamma\delta^2 a_0^3 + 2\beta\gamma\delta^2 a_{-1} a_0 + 2\beta\gamma\delta^2 a_0^2 b_0 \\ & +4\beta\gamma\delta^2 a_1 b_{-1} a_0 - 3\beta\gamma\delta^2 a_1^2 a_{-1} b_0 - 3\beta\gamma\delta^2 a_1^2 a_0 b_{-1} + 4\beta\gamma\delta^2 a_{-1} a_1 b_0 - 3\beta\delta^2 a_1^2 a_{-1} b_0 = 0,\end{aligned}$$

$$\begin{aligned}C_0 = {} & 2\beta\gamma\delta^2 a_1 b_0^2 a_{-1} - 3\beta\gamma\delta^2 a_1 a_0^2 b_{-1} - 4\mu^2 a_{-1}^2 - 3\beta\gamma\delta^2 a_1^2 a_{-1} b_{-1} \\ & -6\beta\gamma\delta^2 a_1 a_0 a_{-1} b_0 + \beta\gamma\delta^2 a_1^2 b_{-1}^2 - 3\beta\gamma\delta^2 a_0^2 a_{-1} + 8\mu^2 a_1 b_{-1} a_{-1} + \beta\gamma\delta^2 a_{-1}^2 + \\ & 6\beta\delta^2 a_1^2 a_{-1}^2 - 4\mu^2\delta a_1 b_0^2 a_{-1} + \beta\delta^2 a_0^4 - 3\beta\delta^2 a_1 a_{-1}^2 + 12\beta\delta^2 a_1 a_0^2 a_{-1} - 2\mu^2 a_1 b_0 a_0 b_{-1} \\ & +2\beta\gamma\delta^2 a_0^2 b_{-1} - 3\beta\delta^2 a_1^2 a_{-1} b_{-1} - 2\omega\mu\delta a_{-1}^2 + 4\beta\gamma\delta^2 a_{-1} a_0 b_0 + 4\beta\gamma\delta^2 a_1 b_0 a_0 b_{-1} \\ & -\beta\delta^2 a_0^3 b_0 - 3\beta\gamma\delta^2 a_1 a_{-1}^2 + \beta\gamma\delta^2 a_0^2 b_0^2 - \beta\gamma\delta^2 a_0^3 b_0 + 2\omega\mu\delta a_1^2 b_{-1}^2 + 2\mu^2 a_0^2 b_{-1} \\ & -3\beta\delta^2 a_1 a_0^2 b_{-1} + 4\beta\gamma\delta^2 a_1 b_{-1} a_{-1} - 3\beta\delta^2 a_0^2 a_{-1} + 4\mu^2 a_0^2 \delta b_{-1} - 6\beta\delta^2 a_1 a_0 a_{-1} b_0 \\ & -4\omega\mu\delta a_{-1} a_0 b_0 - 4\alpha\mu\delta a_0 a_{-1}^2 - 2\mu^2 a_{-1} a_0 b_0 + 2\mu^2 a_1 b_0^2 a_{-1} + 4\omega\mu\delta a_1 b_0 a_0 b_{-1} \\ & -4\mu^2 a_1^2 b_{-1}^2 - 4\alpha\mu\delta a_0^2 a_{-1} + 4\alpha\mu\delta a_1 a_0^2 b_{-1} + 4\alpha\mu\delta a_1^2 a_{-1} b_{-1} = 0,\end{aligned}$$

$$C_{-1} = 12\beta\delta^2 a_1 a_0 a_{-1}^2 + 4\beta\delta^2 a_0^3 a_{-1} - 6\mu^2\delta a_1 b_{-1} a_{-1} b_0 - \mu^2\delta a_1 b_{-1}^2 a_0 - \beta\delta^2 a_0^3 b_{-1}$$
$$+4\mu^2 a_1 b_{-1} a_{-1} b_0 + 6\mu^2\delta a_{-1} a_0 b_{-1} - \alpha\mu\delta a_0^2 a_{-1} b_0 + \alpha\mu\delta a_0^3 b_{-1} + 6\alpha\mu\delta a_1 a_{-1} a_0 b_{-1}$$
$$-4\mu^2 a_1 b_{-1}^2 a_0 - 4\mu^2 a_{-1}^2 b_0 + 4\mu^2 a_{-1} a_0 b_{-1} - 2\omega\mu\delta a_{-1} a_0 b_{-1} - 6\beta\delta^2 a_1 a_{-1} a_0 b_{-1}$$
$$-\mu^2\delta a_{-1} b_0^2 a_0 + \mu^2\delta a_0^2 b_{-1} b_0 + 2\omega\mu\delta a_1 b_{-1} a_{-1} b_0 - 3\omega\mu\delta a_{-1}^2 b_0 + \mu^2\delta a_{-1}^2 b_0$$
$$-\alpha\mu\delta a_1 a_{-1}^2 b_0 + 3\omega\mu\delta a_1 a_0 b_{-1}^2 - \omega\mu\delta a_{-1} b_0^2 a_0 + \omega\mu\delta a_0^2 b_{-1} b_0 - 5\alpha\mu\delta a_0 a_{-1}^2$$
$$-3\beta\gamma\delta^2 a_0^2 a_{-1} b_0 + 4\beta\gamma\delta^2 a_1 a_{-1} b_{-1} b_0 - 3\beta\delta^2 a_1 a_{-1}^2 b_0 - 3\beta\delta^2 a_0 a_{-1}^2 + 2\beta\gamma\delta^2 a_{-1} b_0^2 a_0$$
$$+2\beta\gamma\delta^2 a_1 b_{-1}^2 a_0 - 6\beta\gamma\delta^2 a_1 a_{-1} a_0 b_{-1} - 3\beta\gamma\delta^2 a_0 a_{-1}^2 + 2\beta\gamma\delta^2 a_{-1}^2 b_0 - \beta\gamma\delta^2 a_0^3 b_{-1}$$
$$-3\beta\gamma\delta^2 a_1 a_{-1}^2 b_0 + 2\beta\gamma\delta^2 a_0^2 b_{-1} b_0 + 4\beta\gamma\delta^2 a_{-1} a_0 b_{-1} - 3\beta\delta^2 a_0^2 a_{-1} b_0 = 0,$$

$$C_{-2} = 4\beta\delta^2 a_1 a_{-1}^3 + 6\beta\delta^2 a_0^2 a_{-1}^2 + 2\mu^2 a_{-1} b_0 a_0 b_{-1} - 2\omega\mu\delta a_{-1}^2 b_{-1} + \omega\mu\delta a_0^2 b_{-1}^2$$
$$-\mu^2 a_0^2 b_{-1}^2 - \beta\delta^2 a_{-1}^3 + 2\alpha\mu\delta a_0^2 a_{-1} b_{-1} - 2\alpha\mu\delta a_{-1}^3 - 4\mu^2\delta a_1 b_{-1}^2 a_{-1} - \omega\mu\delta a_{-1}^2 b_0^2$$
$$+4\mu^2\delta a_{-1}^2 b_{-1} + 2\omega\mu\delta a_1 b_{-1}^2 a_{-1} - 2\alpha\mu\delta a_0 b_0 a_{-1}^2 + 2\alpha\mu\delta a_1 a_{-1}^2 b_{-1} - 3\beta\delta^2 * a_0 a_{-1}^2 b_0$$
$$+4\beta\gamma\delta^2 a_{-1} b_0 a_0 b_{-1} + \beta\gamma\delta^2 a_{-1}^2 b_0^2 - 3\beta\gamma\delta^2 a_0^2 a_{-1} b_{-1} - 3\beta\delta^2 a_{-1}^2 a_1 b_{-1} + \beta\delta^2\gamma a_0^2 b_{-1}^2$$
$$+2\beta\gamma\delta^2 a_1 b_{-1}^2 a_{-1} + 2 * \beta\gamma\delta^2 a_{-1}^2 b_{-1} - 3\beta\gamma\delta^2 a_0 a_{-1}^2 b_0 - 3\beta\delta^2 a_0^2 a_{-1} b_{-1}$$
$$-3\beta\gamma\delta^2 a_{-1}^2 a_1 b_{-1} - \beta\gamma\delta^2 a_{-1}^3 - \mu^2 a_{-1}^2 b_0^2 = 0,$$

$$C_{-3} = -\mu^2\delta a_0 b_{-1}^2 a_{-1} - 3\beta\gamma\delta^2 a_{-1}^2 a_0 b_{-1} - \beta\gamma\delta^2 a_{-1}^3 b_0 - \omega\mu\delta a_{-1}^2 b_0 b_{-1}$$
$$+\omega\mu\delta a_0 b_{-1}^2 a_{-1} - \beta\delta^2 a_{-1}^3 b_0 + \beta\gamma\delta^2 a_0 b_{-1}^2 a_{-1} + \alpha\mu\delta a_{-1}^2 a_0 b_{-1} - \alpha\mu\delta a_{-1}^3 b_0$$
$$+2\beta\gamma\delta^2 a_{-1}^2 b_0 b_{-1} + 4\beta\delta^2 a_0 a_{-1}^3 + \mu^2\delta a_{-1}^2 b_0 b_{-1} - 3\beta\delta^2 a_{-1}^2 a_0 b_{-1} = 0,$$

$$C_{-4} = -\beta\delta^2 a_{-1}^3 b_{-1} - \beta\gamma\delta^2 a_{-1}^3 b_{-1} + \beta\gamma\delta^2 a_{-1}^2 b_{-1}^2 + \beta\delta^2 a_{-1}^4 = 0.$$